WEEK 6 NOTES

1. Fourier series

Definition 1.1. Let $f(x)$ denote a function on an interval $(-a, a)$. The *Fourier series* of $\hat{f}$ is defined as

$$\hat{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{a}\right)$$

where the coefficients a_0, a_n and b_n are given by

$$\begin{aligned} a_0 &\equiv \frac{1}{2a}\int_{-a}^{a} f(x)dx, \\ a_n &\equiv \frac{1}{a}\int_{-a}^{a} f(x)\cos\left(\frac{\pi n x}{a}\right)dx, \\ b_n &\equiv \frac{1}{a}\int_{-a}^{a} f(x)\sin\left(\frac{\pi n x}{a}\right)dx. \end{aligned}$$

The key result concerning Fourier series is the following (no proof provided):

Theorem 1.2. *If $f(x)$ is a piecewise smooth function on the interval $(-a, a)$ then the Fourier series $\hat{f}(x)$ of $f(x)$ converges pointwise to:*

(i) *$f(x)$ if $f(x)$ is continuous on $x \in (-a, a)$;*
(ii) *$\frac{1}{2}(f(x_-) + f(x_+))$ if $f(x)$ has a jump at $x \in (-a, a)$.*

Moreover, at the end points $-a$ and a the Fourier series $\hat{f}(x)$ converges to $\frac{1}{2}(f(-a) + f(a))$.

Note. *One observes the following:*

(i) *the interval of definition is symmetric —namely, $[-a, a]$;*
(ii) *if $f(x)$ is an* odd function *on $[-a, a]$ then $a_n = 0$, $n \in \mathbb{N}$;*
(iii) *if $f(x)$ is an* even function *on $[-a, a]$ then $b_n = 0$, $n \in \mathbb{N}$.*

Another important observation is the following: if $f(x)$ in defined on the interval $[0, a]$ we can always extend it to the interval $[-a, a]$ using either *odd* or *even* extensions. In particular, if $f(0) = 0$ then the odd extension of $f(x)$ is continuous and if $f'(0) = 0$ then the even extension of $f(x)$ is smooth.

Some examples illustrate the above general discussion:

Example 1.3. Again, let $x \in (-1, 1)$ and

$$f(x) = x.$$

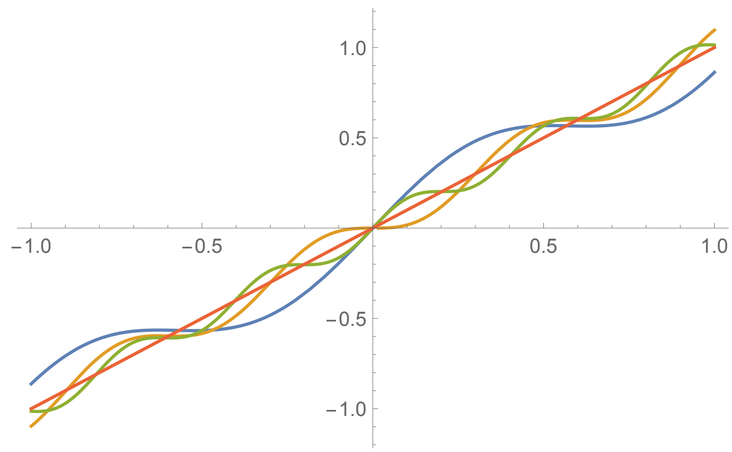
The function $f(x)$ is odd so that $a_n = 0$. For b_n one has that

$$\begin{aligned} b_n &= \int_{-1}^{1} x\sin(n\pi x)dx = 2\int_0^1 x\sin(n\pi x)dx \\ &= -\frac{2x}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx = -\frac{2}{n\pi}(-1)^n + \frac{1}{(n\pi)^2}\sin(n\pi)\Big|_0^1 \\ &= \frac{2(-1)^{n+1}}{n\pi}, \end{aligned}$$

where it has been used that $\cos(n\pi) = (-1)^n$. Hence the Fourier (sine) series is given by

$$\hat{f}(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x).$$

The way $\hat{f}(x)$ approximates $f(x)$ is illustrated in the following diagram:



The blue curve corresponds to a Fourier series with 5 terms, the yellow one with 10 and the green one with 15 terms.

Example 1.4. Let $x \in (-1, 1)$ and

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

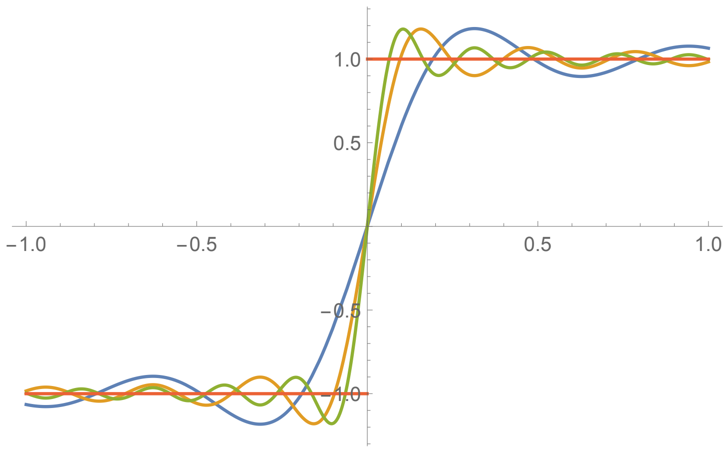
The function $f(x)$ is odd, so that $a_n = 0$, $n = 0, 1, 2, \ldots$. To compute b_n we exploit that $f(x)$ is odd so that

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\sin(\pi n x)dx \\ &= 2\int_0^1 f(x)\sin(\pi n x)dx = 2\int_0^1 \sin(\pi n x)dx \\ &= -\frac{2}{n\pi}\cos(n\pi x)\Big|_0^1 = -\frac{2}{n\pi}\big(\cos(n\pi) - \cos 0\big) \\ &= \frac{2}{n\pi}\big(1 + (-1)^{n+1}\big). \end{aligned}$$

Hence, one has the Fourier (sine) series

$$\hat{f}(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\big(1 + (-1)^{n+1}\big)\sin(n\pi x).$$

The way $\hat{f}(x)$ approximates $f(x)$ is illustrated in the following diagram:



The blue curve corresponds to a Fourier series with 10 terms, the yellow one with 20 and the green one with 30 terms. Observe that at $x = 0$ (the point where f is discontinuous) all the truncated Fourier series pass through the origin —this is consistent with the theory about convergence discussed previously, Theorem 1.2.

The previous two examples from last week were for functions which are odd on $[-1, 1]$. Next is an example with a function which is even.

Example 1.5. Let $x \in [-1, 1]$ and

$$f(x) = \begin{cases} 1 + x & x < 0 \\ 1 - x & x > 0 \end{cases}$$

This function is even so that all the coefficients b_n vanish. For the a_n's one has

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx = \int_0^1 f(x)dx = \int_0^1 (1-x)dx = x\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}.$$

Similarly,

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)dx = 2\int_0^1 f(x)\cos(n\pi x)dx = 2\int_0^1 (1-x)\cos(n\pi x)dx \\ &= 2\int_0^1 \cos(n\pi x)dx - 2\int_0^1 x\cos(n\pi x)dx = \frac{2x}{n\pi}\sin(n\pi x)\Big|_0^1 - \frac{2x}{n\pi}\sin(n\pi x)\Big|_0^1 - \frac{2}{(n\pi)^2}\cos(n\pi x)\Big|_0^1, \\ &= -\frac{2}{(n\pi)^2}\big(\cos(n\pi) - 1\big) = \frac{2}{n^2\pi^2}(1 - (-1)^n), \end{aligned}$$

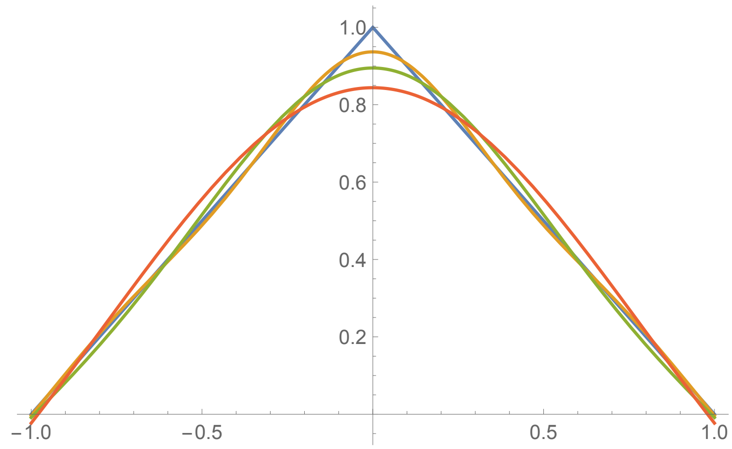
where in the second line above it has been used that

$$\int x\cos(n\pi x)dx = \frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x),$$

which can be readily obtained using integration by parts. Accordingly, the Fourier (cosines) series is given in this case by

$$\hat{f}(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big(1 - (-1)^n\big)\cos(n\pi x).$$

Plots of the series truncated at various orders can be seen in the next figure.



Finally, we conclud ethe discussion of Fourier series with an example of a non-symmetric function showing that it must, necessarily, contain both the sine and cosine series.

Example 1.6. Let Let $x \in [-1, 1]$ and

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$$

In order to compute the Fourier series observe that

$$\int x\cos(n\pi x)dx = \frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x),$$
$$\int x\sin(n\pi x)dx = -\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\sin(n\pi x).$$

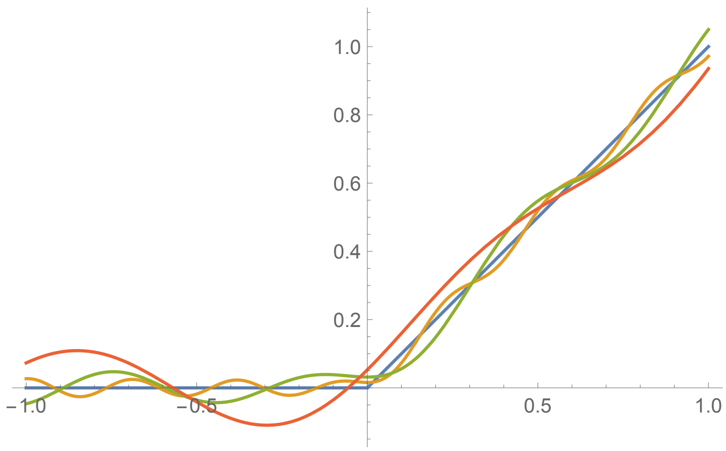
These formulae are readily obtained using integration by parts. Using these formulae one readily finds that

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\int_{-1}^{1} xdx = \frac{1}{4}, \\
a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)dx = \int_{0}^{1} x\cos(n\pi x)dx \\
&= \frac{x}{n\pi}\sin(n\pi x)\Big|_0^1 + \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_0^1 = \frac{1}{n^2\pi^2}\big((-1)^n - 1\big), \\
b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)dx = \int_{0}^{1} x\sin(n\pi x)dx \\
&= -\frac{x}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_0^1 = -\frac{1}{n\pi}(-1)^n = \frac{1}{n\pi}(-1)^{n+1}.
\end{aligned}$$

It follows then that the Fourier series is given by

$$\hat{f}(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \left(\sin(n\pi x) - \frac{1}{n\pi} \cos(n\pi x) \right).$$

Observe that it contains both sine and cosine contributions. Plots of the series truncated at various orders can be seen in the next figure.



2. Example of mixed boundary conditions for initial value problem for wave equations on the interval

Let's see another example of homogeneous wave equations on the interval but with mixed boundary conditions. The strategy is the same as the one we saw last week by separation of variables and change it to an eigenvalue problem. But it will be a variant of the eigenvalue problem we saw last week due to different boundary conditions.

Example 2.1.

$$\begin{cases} U_{tt} = c^2 U_{xx} \\ U_x(0,t) = 0,\ U(\pi,t) = 0 \\ U(x,0) = -\cos\frac{7x}{2}, U_t(x,0) = 0. \end{cases}$$

As a first step, we consider solutions with separated variables of the form $U(x,t) = X(x)T(t)$.

Plugging into the equation, we get

$$\ddot{T} \cdot X = c^2 T \cdot X''$$

So $\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda$ for some constant λ. And we get 2 equations

$$\begin{cases} X'' + \lambda X = 0 & (a) \\ \ddot{T} + c^2 \lambda T = 0 & (b) \end{cases}$$

The equation (a) combined with the boundary conditions give rise to the following eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, X(\pi) = 0. \end{cases} \tag{2.1}$$

Claim 2.2. *We claim again that the eigenvalues $\lambda > 0$.*

Proof of Claim. Multiply both sides of (2.1) by X and integrate from 0 to π we get

$$\int_0^\pi X''(x)X(x)dx + \int_0^\pi \lambda[X(x)]^2 dx = 0$$

$$X'(x)X(x)|_0^\pi - \int [X'(x)]^2 dx + \lambda \int_0^\pi [X(x)]^2 dx = 0$$

$$X'(\pi)X(\pi) - X'(0)X(0) + \lambda \int_0^\pi [X(x)]^2 dx = \int [X'(x)]^2 dx \tag{2.2}$$

Now notice that $X'(\pi)X(\pi) = X'(0)X(0) = 0$ by the boundary conditions and

$$\int_0^\pi [X(x)]^2 dx > 0, \int [X'(x)]^2 dx > 0$$

We must then from (2.2) that $\lambda > 0$. □

Now knowing $\lambda > 0$, we go back to the eigenvalue problem (2.1) and use the theory of constant coefficients ODE that we reviewed in Week 1. the general solutions to (2.1) are

$$X(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x),$$

and its derivative being

$$X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x),$$

The boundary condition $X'(0) = 0$ implies $c_2\sqrt{\lambda} = 0$, namely $c_2 = 0$. Because the solutions are non-trivial, we have $c_1 \neq 0$.

The boundary condition $X(\pi) = 0$ then implies

$$\cos(\sqrt{\lambda}\pi) = 0,$$

giving $\sqrt{\lambda}\pi = \frac{\pi}{2} + n\pi$, for any $n = 1, 2, \ldots$. And the eigenvalues and eigenfunctions are

$$\lambda_n = (\frac{1}{2} + n)^2,$$

$$X_n(x) = \cos[(\frac{1}{2} + n)x].$$

Knowing λ_n, we can also solve (b) and get

$$T_n(t) = a_n \cos[(\frac{1}{2} + n)ct] + b_n \sin[(\frac{1}{2} + n)ct].$$

The general solutions with this boundary condition is then

(2.3)

$$\begin{aligned} U(x,t) &= \sum_{n=1}^\infty X_n(x)T_n(t) \\ &= \sum_{n=1}^\infty a_n \cos[(\frac{1}{2} + n)x]\cos[(\frac{1}{2} + n)ct] + \sum_{n=1}^\infty b_n \cos[(\frac{1}{2} + n)x]\sin[(\frac{1}{2} + n)ct]. \end{aligned}$$

Differentiate with respect to t we get

(2.4)

$$\begin{aligned} &U_t(x,t) \\ =&-(\frac{1}{2}+n)c\sum_{n=1}^{\infty}a_n\cos[(\frac{1}{2}+n)x]\sin[(\frac{1}{2}+n)ct]+(\frac{1}{2}+n)c\sum_{n=1}^{\infty}b_n\cos[(\frac{1}{2}+n)x]\cos[(\frac{1}{2}+n)ct]. \end{aligned}$$

Plug in the initial conditions to (2.3) and (2.4), we have

$$-\cos\frac{7x}{2}=\sum_{n=1}^{\infty}a_n\cos[(\frac{1}{2}+n)x]$$
$$0=(\frac{1}{2}+n)c\sum_{n=1}^{\infty}b_n\cos[(\frac{1}{2}+n)x].$$

This tells us that $b_m = 0$ for any $m = 1, 2\cdots$. Multiply the first identity above by $\cos[(\frac{1}{2}+m)x]$ and integrate from 0 to π we get

$$\int_0^{\pi}-\cos\frac{7x}{2}\cos[(\frac{1}{2}+m)x]dx=\sum_{n=1}^{\infty}\int_0^{\pi}a_n\cos[(\frac{1}{2}+n)x]\cos[(\frac{1}{2}+m)x]dx.$$

Recall from Proposition 3.1 in Week 5 notes that

$$\int_0^{\pi}a_n\cos[(\frac{1}{2}+n)x]\cos[(\frac{1}{2}+m)x]dx=\begin{cases}\frac{2}{\pi}, & n=m\\ 0, & n\neq m.\end{cases}$$

We have the only non-zero $a_m = 0$ unless $(\frac{1}{2}+m)=\frac{7}{2}$. Namely $a_m = 0$ for $m \neq 3$ and

$$-\frac{2}{\pi}=a_3\cdot\frac{2}{\pi}$$
$$a_3=1.$$

Combining all the computation above, the solution to this mixed boundary initial value problem for wave equation on the interval is

$$U(x,t)=-\cos(\frac{7}{2}x)\cos(\frac{7}{2}ct).$$

3. Inhomogeneous wave equations: wave equations with a source

We consider in this section the following initial value problem for inhomogeneous wave equation on the real line $\mathbb{R}$.

(3.1)
$$\begin{cases}U_{tt}-c^2U_{xx}=\psi(x)\\ U(x,0)=f(x)\\ U_t(x,0)=g(x).\end{cases}$$

This is the mathematical model for the evolution of a vibrating string with a source of external force acting on it.

We will first apply the **Principle of Superposition** from the notes of Week 1.

Consider the following 2 equations:

(3.2)
$$\begin{cases}V_{tt}-c^2V_{xx}=0\\ V(x,0)=f(x)\\ V_t(x,0)=g(x),\end{cases}$$

and

$$\begin{cases} W_{tt} - c^2 W_{xx} = \psi(x) \\ W(x,0) = 0 \\ W_t(x,0) = 0. \end{cases} \tag{3.3}$$

We observe that: if V is a solution to (3.2) and W is a solution to (3.3), then $U = V + W$ is a solution to (3.1).

By the D'Almbert's formula from Week 4, we get a solution V to (3.2) by

$$V(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

So we are left to solve (3.3), whose solution is given by the following Theorem called the Duhamel's Principle.

Theorem 3.1 (Duhamel's Principle)**.** *If $\tilde{W}$ is a solution to the equation*

$$\begin{cases} \tilde{W}_{tt} - c^2 \tilde{W}_{xx} = 0 \\ \tilde{W}(x,0) = 0 \\ \tilde{W}_t(x,0) = \psi(x), \end{cases} \tag{3.4}$$

then

$$W(x,t) = \int_0^t (\tilde{W}(x,t-s))ds \tag{3.5}$$

is a solution to (3.3)

Proof. We differentiate W with respect to t once and get

$$\begin{aligned} \frac{\partial}{\partial t} W(x,t) =& \frac{\partial}{\partial t}\int_0^t (\tilde{W}(x,t-s))ds \\ =& \tilde{W}(x,t-s)|_{s=t} + \int_0^t (\tilde{W}_t(x,t-s))ds \\ =& \tilde{W}(x,0) + \int_0^t (\tilde{W}_t(x,t-s))ds \\ =& \int_0^t (\tilde{W}_t(x,t-s))ds. \end{aligned}$$

Differentiating with respect to t again, we get

$$\begin{aligned} \frac{\partial^2}{\partial t^2} W(x,t) =& \frac{\partial}{\partial t}\int_0^t (\tilde{W}_t(x,t-s))ds \\ =& \tilde{W}_t(x,t-s)|_{s=t} + \int_0^t (\tilde{W}_{tt}(x,t-s))ds \\ =& \tilde{W}_t(x,0) + \int_0^t (\tilde{W}_{tt}(x,t-s))ds \\ =& \psi(x) + \int_0^t (\tilde{W}_{tt}(x,t-s))ds. \end{aligned}$$

Similarly, differentiating with respect to x twice, we get

$$\begin{aligned}\frac{\partial^2}{\partial x}W(x,t) =& \frac{\partial^2}{\partial x^2}\int_0^t (\tilde{W}(x,t-s))ds \\ =& \int_0^t (\tilde{W}_{xx}(x,t-s))ds.\end{aligned}$$

So combining these, we get

$$\begin{aligned}W_{tt} - c^2 W_{xx} =& \psi(x) + \int_0^t (\tilde{W}_{tt}(x,t-s))ds - \int_0^t (\tilde{W}_{xx}(x,t-s))ds \\ =& \psi(x) + \int_0^t [\tilde{W}_{tt} - c^2\tilde{W}_{xx}](x,t-s)ds \\ =& \psi(x),\end{aligned}$$

where we used that $\tilde{W}$ satisfies $\tilde{W}_{tt} - c^2\tilde{W}_{xx} = 0$.

Moreover, at time $t = 0$, the initial values of W satisfy

$$\begin{aligned}W(x,0) =& \int_0^0 (\tilde{W}(x,t-s))ds = 0 \\ W_t(x,0) =& \int_0^0 (\tilde{W}_t(x,t-s))ds = 0.\end{aligned}$$

□

On the other hand, we know the solution to (3.4) is given by D'Alembert's formula as follows

$$\tilde{W}(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(r)dr.$$

So, using the Duhamel's principle, we have

$$W(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \psi(r)drds.$$

Combining with our earlier observation using the principle of superposition, the solution to (3.1) is then

(3.6)

$$\begin{aligned}U(x,t) =& V(x,t) + W(x,t) \\ =& \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \psi(r)drds.\end{aligned}$$

Example 3.2. Solve the following inhomogeneous wave equation on the real line.

$$\begin{cases} U_{tt} - c^2 U_{xx} = \cos x \\ U(x,0) = -1 \\ U_t(x,0) = 1 \end{cases}$$

We can apply the formula (3.6) with $\psi(x) = \cos x, f(x) = -1, g(x) = 1$.

$$\begin{aligned}
U(x,t) =& \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \psi(r)drds \\
=& \frac{1}{2}[-1-1] + \frac{1}{2c}\int_{x-ct}^{x+ct} 1ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos r drds \\
=& -1 + \frac{2ct}{2c} + \frac{1}{2c}\int_0^t [\sin(x+ct-cs) - \sin(x-ct+cs)]ds \\
=& -1 + t + \frac{1}{2c}\frac{1}{-c}[-\cos(x+ct-cs)]|_0^t - \frac{1}{2c}\frac{1}{c}[-\cos(x-ct+cs)]|_0^t \\
=& -1 + t + \frac{\cos x}{c^2} - \frac{\cos(x+ct)+\cos(x-ct)}{2c^2}
\end{aligned}$$