QUEEN MARY, UNIVERSITY OF LONDON

MTH6102: Bayesian Statistical Methods

Exercise sheet 4

2023-2024

The deadline for submission is **Monday the 30th October at 11am**. Late submissions receive zero marks. You can submit a Word document, pdf or a clearly legible image of hand-written work.

1. **25 points.** Suppose that the number of new cases of a medical condition observed each week can be modelled using a negative binomial distribution with parameters q and r, q is unknown, while r is known.

   [See the table of common distributions in the Module Content section on QMPlus for details of the negative binomial distribution.]

   We observe n weeks' worth of data, and the number of cases each week was $y_1, \dots, y_n$.

   (a) Show that a beta distribution provides a conjugate prior distribution for q, and find the posterior distribution with such a prior.

   The column in the exercise 2 dataset labelled `y`, contains the observed data $y_1, \dots, y_n$. Assume that r is equal to 3.

   (b) With a uniform prior distribution for q on the interval $[0, 1]$, what is the posterior distribution for q (including the numerical value of the parameters)?

   (c) What is the posterior mean?

   (d) Use R to find the posterior median and a 95% credible interval for q.

2. **25 points.** For the binomial model considered in the lectures, with success probability q and observed data k successes out of n trials, assume a $\text{Beta}(\alpha, \beta)$ prior distribution for q. Show that the posterior mean for q is always in between the prior mean for q and the maximum likelihood estimator $\hat{q}$.

   Show that if the prior is uniform, then the posterior variance for q is always less than the prior variance. Find an example of α, β, k and n where the posterior variance is greater than the prior variance.

3. **25 points.** The number of offspring X in a certain population has probability mass function

$$p(x \mid \theta, \psi) = \begin{cases} \theta & x = 0 \\ (1-\theta)\psi(1-\psi)^{x-1} & x = 1, 2, \dots \end{cases}$$

   where θ and ψ are unknown parameters in the interval $[0, 1]$.

   Write down the likelihood when r zeroes and $n - r$ non-zero values $x_1, x_2, \dots, x_{n-r}$ are observed from n independent observations on X.

Suppose θ and ψ have independent prior beta densities with parameters a, b and c, d, respectively. Show that θ and ψ have independent posterior beta distributions and identify the posterior parameters.

[*Hint:* You may use the fact that two continuous random variables X and Y are independent, if their joint density $f_{X,Y}(x, y)$ can be written as the product of their marginal densities $f_X(x)$ and $f_Y(y)$, respectively. That is, $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.]

4. **25 points.** Your friend transmits an unknown value θ to you over a noisy channel. The noise is normally distributed with mean 0 and a known variance 4, so the value x that you receive is modeled by $N(\theta, 4)$. Based on previous communications, your prior on θ is $N(5, 9)$.

   (a) Suppose your friend transmits a value to you that you receive as $x = 6$. Show that the posterior pdf for θ is $N(74/13, 36/13)$. For this problem, you need to derive the posterior by carrying out the calculations from scratch.

   (b) Suppose your friend transmits the same value θ to you $n = 4$ times. You receive these signals plus noise as $x_1, \ldots, x_4$ with sample mean $\bar{x} = 6$. Using the same prior and known variance σ^2 as in part (a), show that the posterior on θ is $N(5.9, 0.9)$. Plot the posterior and posterior on the same graph. Describe how the data changes your belief about the true value of θ. For this question, you may use the normal updating formulas.

   (c) How do the mean and variance of the posterior change as more data is received?

   (d) IQ in the general population follows a $N(100, 15^2)$ distribution. An IQ test is unbiased with a normal variance of 10^2; that is, if the same person is tested multiple times, their measured IQ will differ from their true IQ according to a normal distribution with 0 mean and variance 100.

      i. Tommy Vard scored an 80 on the test. What is the expected value of his true IQ?

      ii. Anna Taft scored a 150 on the test. What is the expected value of her true IQ?