# Essential Foundation Mathematical Skills

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# Overview

# ILOs

→ Today's lecture is on **GCD and LCM**;

After today's lecture, you are expected to understand the concepts of Greatest Common Divisor and Least Common Multiple of two positive integer numbers.

# Introduction

**History**: GCD and LCM (Greatest Common Divisor and Least Common Multiple).

- The legendary Greek Mathematician Euclid suggested an algorithm for finding the greatest common divisor of two integers, which was later named the **Euclidean algorithm**.
- This algorithm, as well as computationally refined versions, are in widespread use today.

Figure 1: Euclid (ca. 325-270 BC).

# GCD and LCM

### Definition (Greatest Common Divisor)

**Greatest Common Divisor** (**GCD**) or Highest Common Factor (HCF) of two positive integers is the largest positive integer that divides both numbers without remainder.

Examples:

$\gcd(2,4) = 2$
$\gcd(4,6) = 2$
$\gcd(0,5) = 5$

$\gcd(0,0)$ is undefined, so $\gcd(0,0) = 0$

### Definition (Relatively prime)

Two integers $a$ and $b$ are called **relatively prime** if $\gcd(a,b) = 1$.

### Definition (Least Common Multiple)

**Least Common Multiple** (**LCM**) of two integers is the smallest integer that is a multiple of both numbers.

# GCD and LCM

Given two non-negative integers $a$ and $b$, we can calculate $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$ as follows:

1. Perform prime factorization to both integers.
2. Write down all the exponents and then insert any missing prime, raised to the power of 0.
3. The GCD and LCM are the products of **all** primes involved in the prime factorization, each raised to the **smallest** and the **largest** of the two exponents, respectively.

For example, let $a = 270$ and $b = 252$.
Then:

1. $a = 2 \cdot 3^3 \cdot 5$
   $b = 2^2 \cdot 3^2 \cdot 7$
2. $a = 2^1 \cdot 3^3 \cdot 5^1 \cdot 7^0$
   $b = 2^2 \cdot 3^2 \cdot 5^0 \cdot 7^1$
3. $\gcd(a, b) = 2^1 \cdot 3^2 \cdot 5^0 \cdot 7^0 = 2 \cdot 9 \cdot 1 \cdot 1 = 18.$
   $\operatorname{lcm}(a, b) = 2^2 \cdot 3^3 \cdot 5^1 \cdot 7^1 = 4 \cdot 27 \cdot 5 \cdot 7 = 3780.$

$\rightarrow$Note that for integers $a$ and $b$, we have:

$$\gcd(a, b) = \frac{a \cdot b}{\operatorname{lcm}(a,b)}$$

or, equivalently,

$$\operatorname{lcm}(a, b) = \frac{a \cdot b}{\gcd(a,b)}.$$

# GCD and LCM

Important properties:

- If $a$ divides $b$, then $\gcd(a, b) = a$
  and $\operatorname{lcm}(a, b) = b$.

  For example, 3 divides 21 and we have: $\gcd(3, 21) = 3$, $\operatorname{lcm}(3, 21) = 21$.

- The least common multiple of two relatively prime integers is equal to their product.
  $\rightarrow$In other words:
  If $\gcd(a, b) = 1$, then $\operatorname{lcm}(a, b) = a \cdot b$.

- Two distinct primes are necessarily relatively prime,
  but two integers can be relatively prime without any of them being prime.
  $\rightarrow$In other words:
  $a$, $b$ are prime $\implies$ $a$, $b$ are relatively prime.
  $a$, $b$ are relatively prime $\not\Longrightarrow$ $a$, $b$ are prime.

  For example, $\gcd(4, 9) = 1$, but neither 4 or 9 are primes.

# Examples and exam-style questions

**Question**: If $\gcd(x, 12) = 6$ and $\text{lcm}(x, 12) = 180$, then what is the value of $x$?

**Question**: If $\gcd(90, x) = 6$ and $\text{lcm}(90, x) = 180$, then what is the value of $x$?

**Question**: Calculate the GCD and LCM of the following pairs of larger numbers: (48, 60), (72, 90), (81, 108), (54, 135), (112, 168), (104, 156).

**Challenge**: Calculate the GCD and LCM of 40, 168 and 196.

**Question**: Compute the GCD of the following pairs of integers:
$(9^9, 141^2)$, $(66, 18^5)$, $(26^4, 2600)$.

**Question**: Compute the GCD and LCM of the following pairs of integers:
$(48, 50)$, $(56, 44)$, $(236, 168)$.

# Examples and exam-style questions

**HCF AND LCM**
**EXAM-TYPE QUESTIONS**

**NO CALCULATOR**

| | | | |
|---|---|---|---|
| **A1**<br>Express 204 as a product of its prime factors.<br>Show your working clearly. | **A2**<br>Write 792 as a product of its prime factors.<br>Show your working clearly. | **A3**<br>$1400 = 2^p \times 5^2 \times 7$<br>Find the value of $p$. | **A4**<br>Given that $120 = 2^3 \times 3 \times 5$<br>And that $n = 120 \times 108$<br>Write $n$ as a product of powers of its prime factors. |
| **B1**<br>Find the highest common factor (HCF) of 90 and 252 | **B2**<br>Find the lowest common multiple (LCM) of 24 and 42 | **B3**<br>Find the highest common factor (HCF) and lowest common multiple (LCM) of 168 and 180 | **B4**<br>Find the highest common factor (HCF) and lowest common multiple (LCM) of 72, 180 and 540 |
| **C1**<br>$A = 2^3 \times 3 \times 5^2$<br>$B = 2^2 \times 3$<br>Find the HCF and LCM of $A$ and $B$. | **C2**<br>$M = 2^4 \times 3^2 \times 7$<br>$N = 2^2 \times 3^2 \times 5$<br>Find the HCF and LCM of $M$ and $N$. | **C3**<br>$2520 = 2^3 \times 3^2 \times 5 \times 7$<br>$3024 = 2^4 \times 3^3 \times 7$<br>Find the HCF & LCM of 2520 and 3024. Write your answer as a product of prime factors. | **C4**<br>$740\,880 = 2^4 \times 3^3 \times 5 \times 7^3$<br>$980\,100 = 2^2 \times 3^4 \times 5^2 \times 11^2$<br>Find the highest common factor (HCF) of 740 800 and 980 100 |