WEEK 2 NOTES

1. First order PDE with constant coefficients (continued)

Let's continue on solving the first order linear PDEs of the form

$$aU_x + bU_y = 0, \tag{1.1}$$

with $a,\ b \neq 0$ some constants.

1.1. **Further discussion about the geometric approach.** Recall from last week that we know the solution U is constant on each of the following straight lines:

$$y = \frac{b}{a}x + c, \qquad c \quad \text{a constant.}$$

Or equivalently, the straight line equations are given by

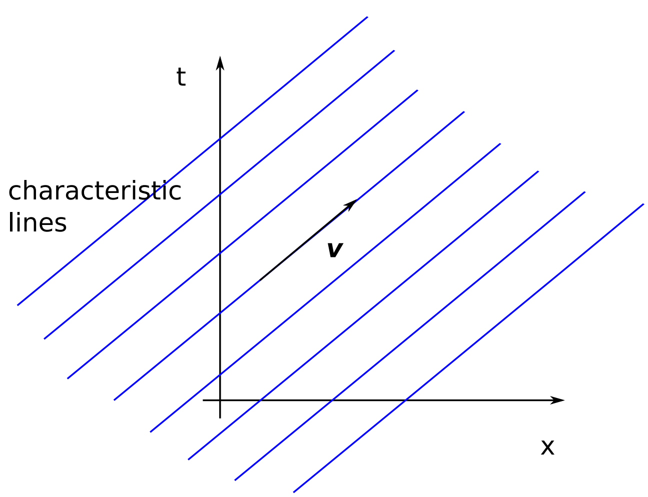
$$bx - ay = c. \tag{1.2}$$

We call these lines *characteristic lines*.

Given that, the solution $U(x,y)$ depends on the value of c only and one can write

$$U(x,y) = f(c) = f(bx - ay)$$

where in the last equality we have used (1.2) and f is, again, an arbitrary function of its argument. Observe that the result we have obtained coincides with what we had using the method 1 (analytic approach).



Question. *How do we specify the function f?*

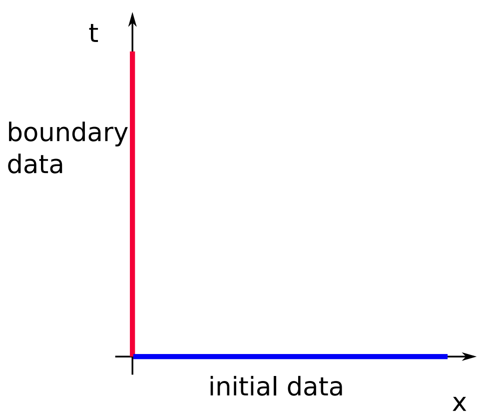
For this, one needs to impose *initial* and/or *boundary conditions*. What are these?

Notation. *In what follows it will be conceptually convenient to use coordinates (x,t) rather (x,y) and think of t as a time —that is, the equations we will analyse describe some process of evolution in time. Conventionally, the time coordinate is assigned to the y-axis.*

1.2. **Initial and boundary conditions.**

Definition 1.1.

i. A prescription of the value of the solution to a pde at $t=0$ (i.e. along the x-axis) will be called an *initial condition.*
ii. A prescription of the value of the solution to a pde at $x=0$ (i.e. along the t-axis) will be called a *boundary condition*.



Note. *More generally, boundary conditions can be prescribed on any line parallel to the t-axis —i.e. lines of the form $x=x_\bullet$ with $x_\bullet$ a constant. More generally, one can have combinations of boundary and initial data. Initial and boundary data arise from physical, geometric and/or commonsensical considerations.*

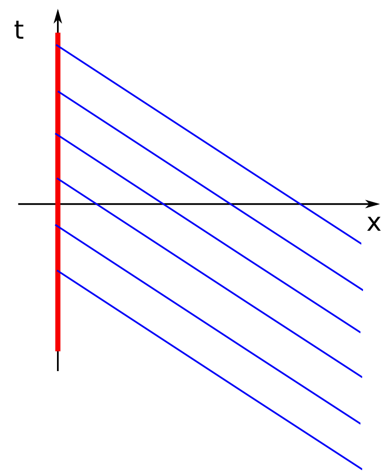
We exemplify these concepts with a couple of examples.

Example 1.2. Solve

$$\begin{aligned} &4U_x - 3U_t = 0, \\ &U(0,t) = t^3 \qquad \text{(boundary conditions).} \end{aligned}$$

From the previous discussion one has that $a=4$, $b=-3$ and the solution to the equation is constant along the lines

$$-3x - 4t = c.$$



Thus, the solution is of the form

$$U(x,t) = f(c) = f(-3x - 4t).$$

We now make use of the boundary condition to determine the function f. On the t-axis one has that $x = 0$ so that $c = -4t$. Thus, one can write $t = -c/4$. Using the latter one can write, on the one hand, that

$$U(0,t) = t^3 = -\frac{1}{64}c^3.$$

On the other hand, from the general solution one has that

$$U(0,t) = f(c).$$

Hence, one concludes that

$$f(c) = -\frac{1}{64}c^3.$$

Thus, the solution determined by the prescribed boundary data is given by

$$U(x,t) = \frac{1}{64}(3x + 4t)^3.$$

One can verify that the above expression is indeed a solution to the original problem by direct evaluation.

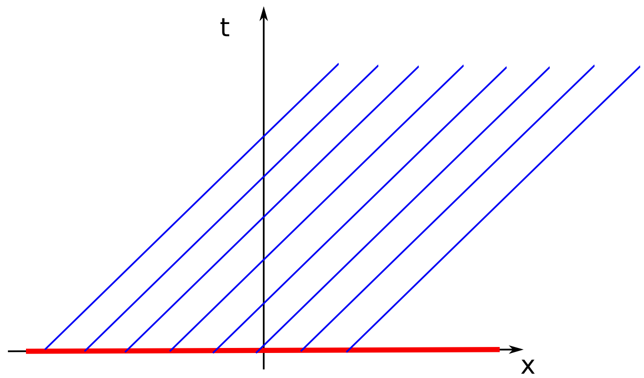
Note. *Observe that after prescribing boundary data one has obtained a* unique *solution.*

Example 1.3. Solve

$$\begin{aligned} &3U_x + 2U_t = 0, \\ &U(x,0) = \sin x \qquad \text{(initial condition)}. \end{aligned}$$

In this case, following the general discussion gives $a = 3$, $b = 2$ so that the solution is constant along the line

$$2x - 3t = c.$$



The general solution is then given by

$$U(x,t) = f(c) = f(2x - 3t).$$

The characteristic lines intersect the x-axis at $x = c/2$ (i.e. $c = 2x$). Now,

$$U(x,0) = \sin x = \sin \frac{c}{2}.$$

However, one also has that

$$U(x,0) = f(c).$$

Accordingly, one concludes that

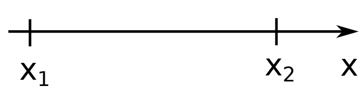
$$f(c) = \sin \frac{c}{2},$$

and the solution for the given initial data is given by

$$U(x,t) = \sin \frac{1}{2}(2x - 3t).$$

1.3. **An application: traffic models.** Equations like (1.1) arise in many models. In this section we consider a traffic model. This theory was invented in Manchester by Sir J. Lighthill and G. B. Whitham in 1955. The ideas of this model are also applicable in the discussion of glacier flows and sedimentation in river deltas.

In what follows we are interested in describing the traffic along a one-directional road. We assume the road to be straight —although this is not key for the discussion. A position along the road is described by the coordinate x. In this model the *traffic density* $\rho(x,t)$ is defined as the number of cars (or other vehicles) per unit distance at time t and position x. The traffic density is a type of average.



The problem one is interested in solving is to find $\rho(x,t)$ assuming that the initial density $\rho(x,0)$ is known —that is, we know the initial distribution of cars along the road.

We construct an equation for the model by the following considerations: the number of cars between two (arbitrary) fixed points x_1 and another x_2 at time t is given by

$$\int_{x_1}^{x_2} \rho(x,t)dx.$$

The rate of change of the number of cars between x_1 and x_2 with respect to time is given by

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(x,t)dx = \int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho(x,t)dx \\ &\approx \text{(Rate of change of number of cars at } x_1 \text{ at time } t) \\ &\qquad -\text{(Rate of change of number of cars at } x_2 \text{ at time } t) \\ &= Q(x_1,t) - Q(x_2,t).\end{aligned}$$

We call $Q(x,t)$ the *flow* at the position x and time t. The above expression says in plain words that the change in the number of cars between x_1 and x_2 is due to cars entering and leaving the section of the road under consideration —we can call this the *law of conservation of cars*. To complete the model one needs to say something about the flow $Q(x,t)$. A reasonable assumption is that when the density of cars is very small (i.e. $\rho << 1$) then Q is proportional to the density of the cars, namely $Q(\rho) \approx k\rho$ with $k > 0$ a constant —cf. the plot of $Q(\rho)$.

Using the Fundamental Theorem of Calculus (keeping t fixed) one has that

$$Q(x_2,t) - Q(x_1,t) = \int_{x_1}^{x_2} \frac{\partial}{\partial x}Q(x,t)dx.$$

Thus, one has that

$$\int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho(x,t)dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x}Q(x,t)dx.$$

However, the points x_1 and x_2 defining the section of the road under consideration are arbitrary so

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}Q(x,t).$$

Now, to conclude, we need to compute $\partial Q(x,t)/\partial x$ given that $Q(x,t) = Q(\rho)$. For this we use the chain rule:

$$\frac{\partial Q}{\partial x}(\rho) = \frac{d}{d\rho}Q\frac{d\rho}{dx} = Q'\rho_x.$$

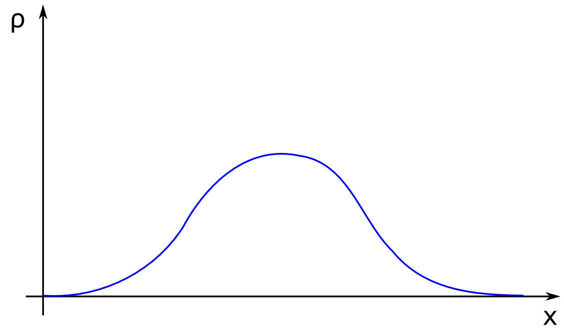
Hence, one obtains the equation

$$\rho_t + Q'(\rho)\rho_x = 0.$$

On the other hand, by our assumption above, $Q' = k$, so one obtains

$$\rho_t + k\rho_x = 0. \tag{1.3}$$

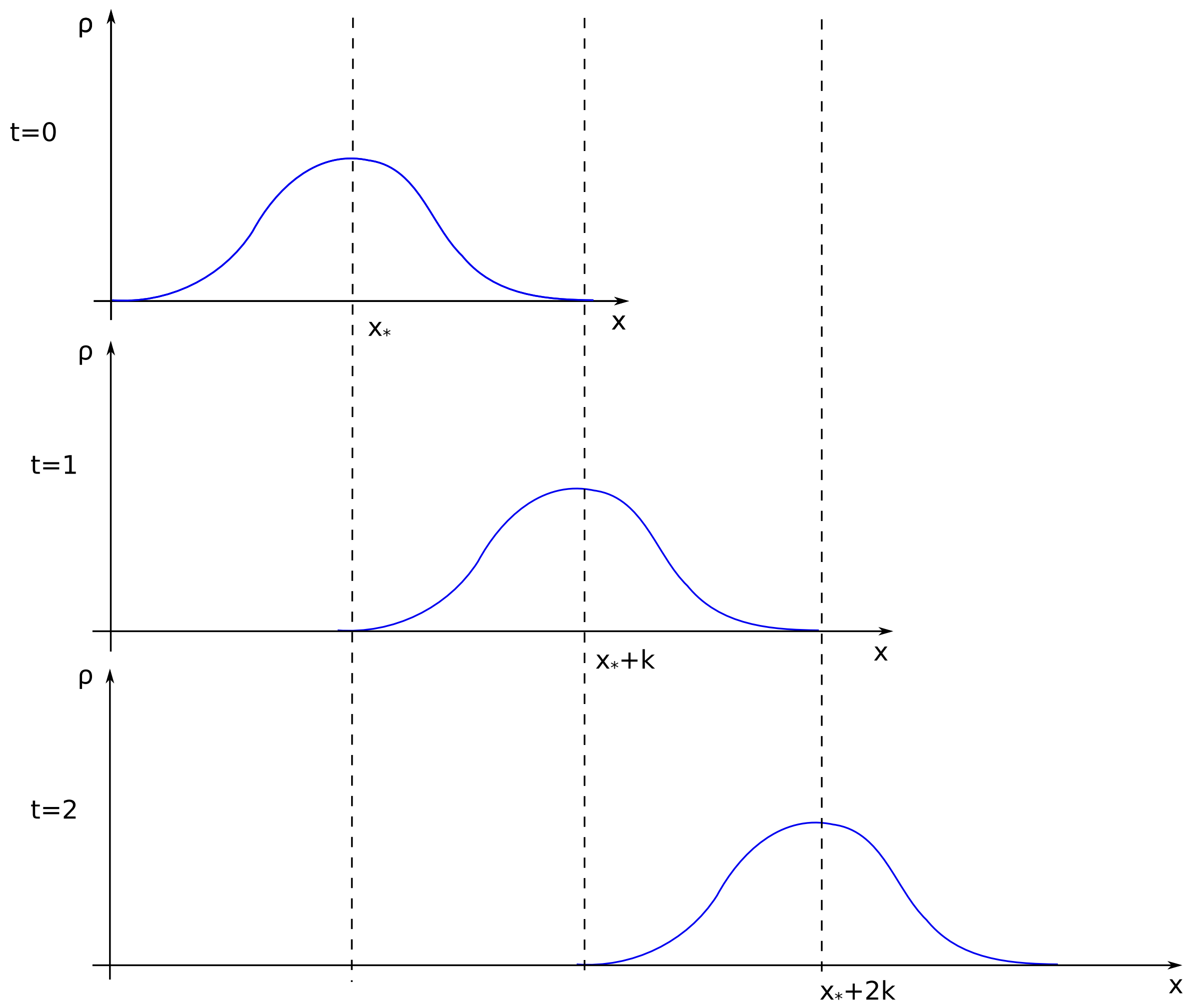
This is an equation of the same form as (1.1) —so, we know how to solve it! Now, assume that one is given an initial density of cars $\rho(x,0) = f(x)$ having the form of a bump as in the figure below:



Using the method discussed in the previous sections we find that the solution for the given initial data is given by

$$\rho(x,t) = f(x - kt).$$

The interpretation of this solution is as follows: as time increases, the initial bump moves to the right (keeping its shape) —see the figure below. In other words, if there is little traffic in the road, the cars move basically in formation.



2. The general linear first order PDE with variable coefficients

In this section we will discuss how to solve the equation

$$a(x,y)U_x + b(x,y)U_y = c(x,y)U + d(x,y) \tag{2.1}$$

where a, b, c and d are functions of the coordinates (x,y). The *method of characteristics* used to analyse the equation with constant coefficients can be extended to consider this type of equation. The point of departure is the geometric perspective we followed in the previous section.

2.1. **Geometric approach.**

Equation (2.1) can be written as

$$\big(a(x,y), b(x,y)\big) \cdot \nabla U = c(x,y)U + d(x,y)$$

so that

$$\nabla_{\vec{v}} U = c(x,y)U + d(x,y), \qquad \text{with} \quad \vec{v} \equiv \big(a(x,y), b(x,y)\big).$$

In this case the vector $\vec{v}$ is no longer constant. This means that the characteristics are no longer lines but curves. If the characteristics are of the form $y = y(x)$ then they satisfy the ordinary differential equation

$$\frac{dy}{dx} = \frac{b(x, y(x))}{a(x, y(x))}. \tag{2.2}$$

Question. *Can an ode always be solved?*

There is a general result known as the **Picard-Lindelöf theorem** that basically says that an ode always has a solution for given initial conditions. Of course, this result does not say (at least directly) how to solve the equation.

In what follows we assume that we can solve the ode (2.2). This gives the solution in the form $y = y(x)$. Now, we compute the derivative of U along the characteristics —for this we use the chain rule as follows:

$$\begin{aligned}\frac{d}{dx}U(x, y(x)) &= \frac{dx}{dx}\frac{\partial U}{\partial x} + \frac{dy}{dx}\frac{\partial U}{\partial y} \\ &= U_x + \frac{b(x, y(x))}{a(x, y(x))}U_y.\end{aligned}$$

Finally, using equation (2.1) one obtains

$$\frac{d}{dx}U(x, y(x)) = \frac{c(x, y(x))}{a(x, y(x))}U + \frac{d(x, y(x))}{a(x, y(x))}. \tag{2.3}$$

This equation is, again, another ode. Its solutions yields the value of $U(x, y)$ along a given characteristic curve.

Note. *If the characteristic curves cover the whole plane $\mathbb{R}^2$, then one obtains a solution to equation* (2.1) *on the whole of $\mathbb{R}^2$. On the other hand, if the characteristics do not exist somewhere, then the solution breaks down there —intuitively, one can say that the solution does not know where to go!*

2.2. **Examples.** We now exemplify the general theory of the previous subsection with a number of examples.

Example 2.1. Find the general solution of

$$U_x + tU_t = 0.$$

The equation can be rewritten as

$$(1, t) \cdot \nabla U = 0,$$

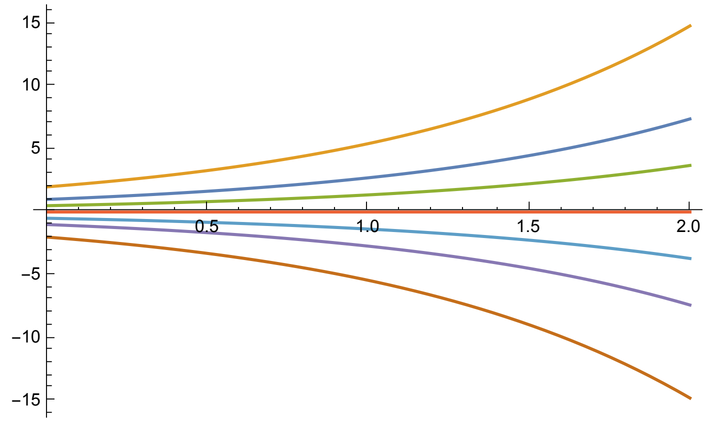
so that U is constant along the curves with tangent given by $\vec{v} = (1, y)$. The slope of the curves is $y/1$ and, hence, the ordinary differential equation to be solved is

$$\frac{dt}{dx} = t.$$

The solutions are given

$$t(x) = Ce^x, \qquad \text{with} \quad C \quad \text{a constant.}$$

These are the characteristics of the pde. A plot for various values of C is given below.



It is observed that, in fact, the whole planes can be covered by these curves by varying C —i.e.

$$\mathbb{R}^2 = \{(x,t) \,|\, t = Ce^x,\ C \in \mathbb{R}\}.$$

Now, observe that for $U(x,t(x)) = U(x,Ce^x)$ one has that

$$\frac{d}{dx}U(x,Ce^x) = U_x + Ce^x U_t$$
$$= U_x + yU_t = 0.$$

Thus, along each characteristic curve the solution is a constant and the solution can only depend on C —that is,

$$U(x,y) = f(C).$$

However, as $t = Ce^x$, one has that $C = te^{-x}$. Hence, one can write the general solution as

$$U(x,t) = f(te^{-x}),$$

where f is an arbitrary function of a single variable.

Example 2.2. Find the solution to the boundary value problem

$$U_x + tU_t = 0,$$
$$U(0,t) = t^3.$$

From the previous example we know that the general solution is given by

$$U(x,t) = f(te^{-x}).$$

Thus, one has that

$$U(0,t) = f(t) = f(C).$$

But one also has that

$$U(0,t) = t^3 = C^3$$

Hence, $f(C) = C^3$ and the required solution is given by

$$U(x,t) = \left(te^{-x}\right)^3 = t^3 e^{-3x}.$$

After obtaining this solution, we will need to check by direct computation that $U(x,t) = t^3e^{-3x}$ is, indeed, the required solution!

Let's next see an example of an inhomogeneous PDE. In this case, the solution is not constant along characteristic curves, but will satisfy an ODE along the characteristic.

Example 2.3. Find the general solutions for $U_x - U_t = 1$.

We see in this case $a = 1, b = -2, c = 0, d = 1$.

Our first step is to find the characteristic curves, they are given by solutions to the ODE

$$\frac{dt}{dx} = -1.$$

Solve it and we get the characteristic (lines) $t = -x + C$.

Now we can write the left hand side of the equation as an ordinary derivative along characteristic lines using $t(x) = -x + c$. Namely

$$\frac{d}{dx}U(x,t(x)) = U_x + U_t \cdot \frac{dt}{dx} = U_x + U_t \cdot (-1) = 1,$$

where we used that U satisfies the PDE in the last equality.

Integrate on both sides with respect to x, we get

$$U(x,t) = \int 1dx + f(C) = x + f(C),$$

There is a dependence on $f(C)$ because the constant may depend on which characteristic line it is on, and C is used to parametrise the family of characteristics.

Now substitute back $C = t + x$ using the characteristic equation, we get the general solution

$$U(x,t) = x + f(t+x),$$

for any f.

Sometimes, the characteristic lines may not be filling up the whole planes. In those cases, the solutions may not exists for every (x,t) on the plane.

Example 2.4. Find the solution to the boundary value problem

$$\sqrt{1-x^2}U_x + U_t = 0,$$
$$U(0,t) = t.$$

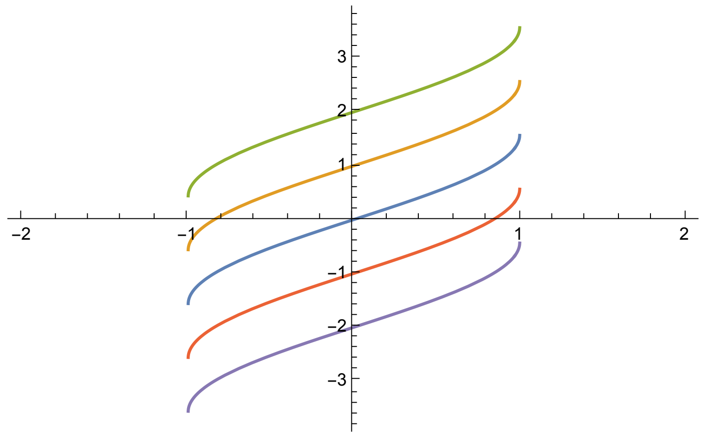
In this case the ode for the characteristic curves is given by

$$\frac{dt}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

The general solution to this ode is

$$t(x) = \arcsin x + C$$

—why? A plot of the curves for various values of C is given below:



Observe, again, that the curves do not cover the whole plane. Now, by the general theory (or direct computation)

$$\frac{d}{dx}U(x,t(x)) = 0,$$

so that

$$U(x,t(x)) = f(C).$$

Hence, the general solution to the equation is given by

$$U(x,t) = f(t - \arcsin x).$$

Evaluating at $x = 0$ one finds that $U(0,t) = f(t)$. Thus, comparing with the boundary condition one concludes that $f(t) = t$. Hence, the solution we look for is

$$U(x,t) = t - \arcsin x.$$