Random Processes – 2023/24 **Problem Sheet 6**

The questions on this sheet are based on the material on first returns, recurrence and transience from Week 5 and 6 lectures.

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1. Consider the Markov chain on state space $S = \{1, 2, 3\}$ with transition matrix

$$\begin{pmatrix} 0 & 1/5 & 4/5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Explain why this chain has a unique equilibrium distribution but no limiting distribution.

(b) For each state $s \in S$ calculate directly the distribution of R_s (the time of first return to s) and $\mathbb{E}(R_s)$.

(c) Use part (b) (and a Theorem from the notes) to write down the equilibrium distribution for the chain.

(d) Check that the answer you get is the same as that obtained by the usual method of solving the appropriate matrix equation.

2. Consider the Markov chain on state space $S = \{0, 1, 2, \dots\}$ with transition probabilities $p_{i,i+1} = \frac{i+1}{i+2}$, $p_{i,0} = \frac{1}{i+2}$ for $i \geqslant 0$ (all others being 0).

(a) Calculate $f_0^{(t)}$ and f_0 for this chain and deduce from this that state 0 is recurrent.

(b) Calculate $\mathbb{E}(R_0)$ and decide whether state 0 is positive recurrent or null recurrent.

(c) Is state 1000 transient, null recurrent or positive recurrent? Why?

(d) How does this chain compare with the success-runs chain? (Example 23 from the notes)

3. Consider the biased random walk with $S = \mathbb{N}$ with reflecting boundary in which the transition probabilities are

$$\begin{aligned} p_{0,1} &= 1 \\ p_{i,i+1} &= 1/3 \text{ for all } i \geqslant 1 \\ p_{i,i-1} &= 2/3 \text{ for all } i \geqslant 1 \\ p_{i,j} &= 0 \text{ for all other } i, j \end{aligned}$$

(a) Find the unique equilibrium distribution for this chain.

[Hint: Write down the equations for w_j; find an expression for the first few; guess the general solution; check that your guess is indeed a solution.]

(b) Is the chain transient, null recurrent or positive recurrent?

(c) If $X_0 = 0$, what is the expectation of $\min\{t \geqslant 1 : X_t = 0\}$?

4. Let $(X_0, X_1, \ldots)$ be the Markov chain on state space $S = \mathbb{Z}$ and transition probabilities

$$\begin{aligned} p_{i,-i} = p_{i,i-1} &= 1/2 \quad \text{if } i > 0 \text{ is even} \\ p_{i,i-1} = p_{i,i} = p_{i,i+1} &= 1/3 \quad \text{if } i > 0 \text{ is odd} \\ p_{i,i+1} &= 2/3 \quad \text{if } i < 0 \\ p_{i,i-1} &= 1/3 \quad \text{if } i < 0 \\ p_{0,0} &= 1 \end{aligned}$$

(a) Sketch the transition graph.

(b) Identify the communicating classes.

(c) For each communicating class decide whether it is transient, null recurrent or positive recurrent. Justify your answers briefly.

5. [Challenge Question] If $s \in S$ is a state of a Markov chain, we say that s is 2-periodic if $p_{ss}^{(t)} = 0$ for all odd t.

(a) Show that 2-periodicity is a class property (that is if i is 2-periodic and j is a element of the same communicating class as i then j is 2-periodic).

(b) Can you find a Markov chain which contains both a loop (that is a state s with $p_{ss} > 0$) and a 2-periodic state.

(c) Can you find an irreducible Markov chain which contains both a loop and a 2-periodic state?

Some recent exam questions on the material in Week 6 include:

- Main Exam Period 2018. Question 5(c,d,e)
- Main Exam Period 2019. Question 5
- January 2022 Exam. Question 2(a,e)
- January 2023 Exam. Question 1(e,f)

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