Random Processes – 2023/24 Problem Sheet 4

The questions on this sheet are based on the material on limiting and equilibrium from Week 3 and 4 lectures. Question 3 is about the proof of a small result we used in lectures. The other questions are mainly about checking the properties of being irreducible and regular, and computing equilibrium and limiting distributions.

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1. Look back at Question 3 on Problem Sheet 3. For each of the parts (a,b,c,d,e) decide whether the corresponding Markov chain is irreducible, regular, or neither. Which of them have a limiting distribution and what is it?

2. Let $(X_0, X_1, \ldots)$ be the Markov chain on state space $\{1, 2, 3, 4, 5\}$ with transition matrix:

$$\begin{pmatrix} p & 0 & 0 & 0 & 1-p \\ 0 & 0 & 0 & 0 & 1 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

(a) For which values of p is this Markov chain irreducible?

(b) For which values of p is this Markov chain regular?

(c) Find the limiting distribution for this Markov chain when $p = 1/2$ explaining your method carefully.

(d) Suppose that the Markov chain of part (c) represents the position of a robot moving around an environment with 5 regions, where X_i is the region it is in i minutes after being placed in the environment. Describe what the limiting distribution means in this context as you would explain it to a non-mathematician. Your answer should consist of one or two sentences with no mathematical terminology or symbols.

3.

(a) Prove that if an irreducible Markov chain on a finite state space has a state i with $p_{ii} > 0$ then it is regular.

(b) Does the same result hold if we allow the state space to be infinite? Justify your answer.

4. I have n balls distributed between 2 buckets. At each time step I pick one of the balls at random (each being picked with probability $1/n$) and move it from whichever bucket it is currently in to the other bucket.

(a) Describe a Markov chain which models this process.

(b) Give the transition graph and transition matrix when $n = 4$.

(c) Does the Markov chain in part (a) have a limiting distribution?

(d) Does it have an equilibrium distribution?

(e) If your answer to either of (c) or (d) was 'yes', what is that distribution when $n = 4$?

(f) How do your answers to (c) and (d) change if instead of moving our randomly chosen ball, we take it out and put it back in a random bucket (choosing each one with probability $1/2$).

Some recent exam questions on the material in Weeks 3 and 4 include:

- Main Exam Period 2019. Questions 1 and 2
- January 2020 Exam. Question 2(c,d)
- January 2021 Exam. Question 1
- January 2022 Exam. Question 1(b-e)
- January 2023 Exam. Question 3(a-c,e-g)

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