Random Processes – 2023/24 **Problem Sheet 2**

The questions on this sheet are based on the material on the technique of first-step analysis from Week 2 lectures. Questions 1, 2 and 3 are typical examples of first-step analysis. If you have time and enthusiasm then Question 4 is an application which needs a more ingenious approach and Question 5 explores some of the theory.

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1. A Markov chain on state space $\{1,2,3,4\}$ has transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) Which states are absorbing?

   (b) Find the probability that the process ends up in state 1 given that it starts in state 2?

2. A Markov chain on state space $\{1,2,3,4,5\}$ has transition matrix and $X_0 = 1$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4/5 & 1/5 & 0 \\ 0 & 1/6 & 2/3 & 0 & 1/6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) Which states are absorbing?

   (b) Calculate the probability that the process is absorbed at state 4.

   (c) Calculate the expectation of the time of absorption.

   (d) Calculate the expectation of the number of visits to state 2 before absorption.

   (e) Suppose that you lose £5 for each visit to state 2 and gain £10 for each visit to state 3. Use first-step analysis to calculate the expectation of the number of pounds you gain before absorption.

   (f) How could you work out the answer to (e) from (c) and (d) without doing the first-step analysis of part (e)?

3. I have £1 and you have £2. We play the following game. A coin which has probability p of showing heads is tossed. If the coin shows heads then you pay me £1; if it shows tails then I pay you £1. The game stops when one of us has no money left.

(a) Describe a Markov chain which models this process.

(b) For what value of p is the game fair in the sense that the probability that I end up with no money is equal to the probability that you end up with no money?

4. A standard fair 6-sided die is rolled repeatedly until the sum of two consecutive rolls is exactly 4.

(a) Show how to model this process using a Markov chain with 36 states.

(b) Show how to model this process using a Markov chain with a significantly smaller number of states.

(c) Calculate the expectation of the number of rolls the process lasts.

5. [Challenge Question] Find an example of a Markov chain on a finite state space with two absorbing states for which the probability that the process eventually reaches an absorbing state is strictly between 0 and 1. Do the first-step analysis equations for finding the probability of absorption in a particular absorbing state in your chain have a unique solution? If not how would you identify the correct solution in your example?

Some recent exam questions on the material in Week 2 include:

- January 2020 Exam. Question 3
- January 2021 Exam. Question 2
- January 2023 Exam. Question 1(b,c)

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