MTH6134 Statistical Modelling II
Exercises

Autumn 2023

Exercises built upon a list provided by Dr S Coad (formerly of QMUL).

1. Suppose that $Y_i \sim \text{ Bin}(r_i, \pi)$ for $i = 1, 2, \ldots, n$, all independent, where the r_i are known.
    1. Write down the likelihood for the data $y_1, \ldots, y_n$.
    2. Find the maximum likelihood estimator $\hat{\pi}$ of π.
    3. Prove that $\hat{\pi}$ is an unbiased estimator of π.
2. Suppose you have the following binomial data from a single binomial sample: $r = 15, y = 7$.
    1. Write down the likelihood for the data y.
    2. Find the maximum likelihood estimator $\hat{\pi}$ of π.
    3. Using `R`, make a plot of the likelihood function $L(\pi)$. Examine and describe this function.
    4. Consider the following binomial sample: $r = 105, y = 49$. Repeat the computation of the likelihood $L(\pi)$, the maximum likelihood estimate $\hat{\pi}$ and the plot of $L(\pi)$. Compare the results with those of the original data and comment.
3. Consider the following binomial data pairs (r, y): $(60, 19), (70, 25), (30, 15), (40, 14), (20, 9)$.
    1. Repeat the computations of steps 1-3 of the previous question (problem 2). In this case, consider and analyze each data pair separately.
    2. Analyze the data jointly, using the result of the problem 1.
    3. Compare the results of the two analyses. Are the estimates that you obtained related?
4. Suppose that $Y_i \sim \text{Poisson}(\mu)$ for $i = 1, 2, \ldots, n$, all independent.
    1. Write down the likelihood for the data $y_1, \ldots, y_n$.
    2. Find the maximum likelihood estimator $\hat{\mu}$ of μ.
    3. Prove that $\hat{\mu}$ is an unbiased estimator of μ.
5. The following count data $5, 1, 3, 5, 5, 4, 3, 2$, are assumed to be a series of independent realizations of $\text{Poisson}(\mu)$.
    1. Write down the likelihood for the data $y_1, \ldots, y_n$.
    2. Find the maximum likelihood estimator $\hat{\mu}$ of μ.
    3. Plot the likelihood function $L(\mu)$ with `R`. Examine and describe this function.

4. Now suppose that you have a sample of Poisson data with the same sample value $\bar{y}$ as with the data above, but with $n = 16$. Redo the plot of $L(\mu)$, compare with the first plot and comment.

6. Consider the count data $47, 40, 46, 41, 40$. Repeat the computations of items 1-3 of problem 5.

7. Suppose that $Y_i \sim \mathrm{N}(\beta x_i, \sigma^2)$ for $i = 1, 2, \ldots, n$, all independent, where x_i is a known covariate.

   1. Write down the likelihood for the data $y_1, \ldots, y_n$.
   2. Find the maximum likelihood estimators $\hat{\beta}$ and $\hat{\sigma}^2$ of β and σ^2.
   3. Prove that $\hat{\beta}$ is an unbiased estimator of β.

8. In this problem we study properties of the link function $g(u) = log(u)$.

   1. Determine the domain and range of $g(u)$.
   2. For which type of response is the link $g(u)$ function most suitable?
   3. Invert $g(u)$ and compute directly the derivative of the inverse $g^{-1}(u)$, i.e. $\frac{d}{du}g^{-1}(u)$.
   4. Find out about the inverse function theorem. Use the inverse function theorem to compute the derivative of the inverse $g^{-1}(u)$.
   5. Repeat the steps and computations above for the following link functions:
      (a) The identity link $g(u) = u$.
      (b) The inverse quadratic link $g(u) = u^{-2}$.
      (c) The square root link $g(u) = u^{-1/2} = \sqrt{u}$.
      (d) The logit link $g(u) = \log(u/(1-u))$.
      (e) The complementary log-log link $g(u) = \log(-\log(1-u))$.
      (f) The Cauchy link $g(u) = \Phi^{-1}(u)$, where $\Phi(u) = \frac{1}{2} + \frac{1}{\pi}\arctan(u)$.
      (g) **(Medium)** The Gumbel link $g(u) = \Phi^{-1}(u)$, where $\Phi(u) = \exp(-\exp(-u))$ is the cumulative Gumbel distribution. Discuss one potential disadvantage of this link.
      (h) **(Hard)** The probit link $g(u) = \Phi^{-1}(u)$, where $\Phi(u)$ is the cumulative distribution of the standard normal random variable.

9. Suppose that $Y_i \sim \mathrm{N}(\mu, \sigma^2)$ for $i = 1, 2, \ldots, n$, all independent,

   1. Write down the likelihood for the data $y_1, \ldots, y_n$. **Hint**: Try to reuse the equations in lecture notes (ditto for the second item).
   2. Determine analytically the maximum likelihood estimates.
   3. Find the Fisher information matrix.

10. The observations 6.3, 4.2, 6.02, 4.32, 4.04, 3.95 are assumed to be independent realizations of the normal model $\mathrm{N}(\mu, \sigma^2)$.

    1. Using `R`, compute the likelihood estimates with formulæ $\hat{\mu} = \bar{y}$ and $\hat{\sigma}^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2/n$.
    2. Formulate the estimation of μ, σ like a linear regression in `R` and compute the estimates $\hat{\mu}, \hat{\sigma}^2$. In other words, use the function `lm` and process its output.

11. The Federal Trade Commission measured the numbers of milligrammes of tar (x) and carbon monoxide (y) per cigarette for all domestic filtered and mentholated cigarettes of length 100 millimetres. A sample of 12 brands yielded the following data:

Brand	x	y
Capri	9	6
Carlton	4	6
Kent	14	14
Kool Milds	12	12
Marlboro Lights	10	12
Merit Ultras	5	7
Now	3	4
Salem	17	18
Triumph	6	8
True	7	8
Vantage	8	13
Virginia Slims	15	13

1. Calculate the least squares regression line for these data.
2. Plot the points and the least squares regression line on the same graph.
3. Find an unbiased estimate of σ^2.

12. Consider the data on manatees in Practical 1. Use `R` to answer the questions below.

    1. Produce a scatterplot of the data. Does the relationship between y and x seem to be linear?
    2. Fit a simple linear regression model to the data. Give the values of $\hat{\beta}_0$ and $\hat{\beta}_1$, and test $H_0 : \beta_1 = 0$.
    3. By examining the residual plots, comment on whether there is any reason to doubt the assumptions of the model.

13. Suppose that $Y_i \sim \mathrm{N}(\beta x_i, \sigma^2)$ for $i = 1, 2, \ldots, n$, all independent, where x_i is a known covariate.

    1. Find the Fisher information matrix. **Hint**: Try to reuse the equations in the lecture notes.
    2. State the asymptotic distributions of the maximum likelihood estimators $\hat{\beta}$ and $\hat{\sigma}^2$ of β and σ^2.
    3. Explain why the distribution of $\hat{\beta}$ is exact.

14. Consider the manatees' data again and a regression model that passes through the origin.

    1. Explain in simple terms what does a model going through the origin imply for the manatees' data.
    2. Using the data, compute with the help of `R` an estimate of the matrix V and then give its inverse V^{-1} which is an estimation of the variance-covariance matrix for the model parameters.
    3. Briefly comment upon your results.

15. Consider the model $Y_i \sim \mathrm{N}(\mu, \mu^2)$ for $i = 1, 2, \ldots, n$, all independent,

    1. Write down the likelihood for the data $y_1, \ldots, y_n$.
    2. Determine analytically the maximum likelihood estimate.
    3. Find the Fisher information matrix.

16. Suppose that $Y_i \sim \mathrm{N}(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$, all independent, where $\mu_i = \mathbf{x}_i\beta$ and the σ_i are **known**.

    1. Write down the likelihood for the data $y_1, \ldots, y_n$.
    2. Show that $\hat{\beta} = (\mathbf{X}^\top \Sigma^{-1} \mathbf{X})^{-1} \mathbf{X}^\top \Sigma^{-1} \mathbf{Y}$ is the maximum likelihood estimator of β. Here $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_n^2)$.
    3. Find the Fisher information matrix.

17. Consider the following data (-1,3.1), (-1,2.1), (0,5.4), (0,4.2), (1,6), (1,6), which is given as pairs (x_i, y_i). Implement in `R` the results of the model $Y_i \sim \mathrm{N}(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$, all independent, where $\mu_i = \beta_0 + \beta_1 x_i$. The σ_i are known as $\sigma_1^2 = \sigma_2^2 = 1$, $\sigma_3^2 = \sigma_4^2 = 2$, $\sigma_5^2 = \sigma_6^2 = 4$.

    In particular, compute the maximum likelihood estimate $\hat{\beta}$ and its asymptotic variance-covariance matrix.

18. Suppose that $Y_i \sim \mathrm{Bin}(r_i, \pi_i)$ for $i = 1, 2, \ldots, n$, all independent, where the r_i are known, $\pi_i = \beta_0 + \beta_1 x_i$ and x_i is a known covariate.

    1. Write down the likelihood for the data $y_1, \ldots, y_n$.
    2. Obtain the likelihood equations.
    3. Find the Fisher information matrix.

19. Suppose that $Y_i \sim$ Poisson(μ_i) for $i = 1, 2, \ldots, n$, all independent, where $\mu_i = \beta_0 + \beta_1 x_i$ and x_i is a known covariate.

    1. Write down the likelihood for the data $y_1, \ldots, y_n$.
    2. Obtain the likelihood equations.
    3. Find the Fisher information matrix.

20. Consider the data on diabetics in Practical 2. Use `R` to answer the questions below.

    1. Produce scatterplots of y against each of the explanatory variables. Does y appear to be linearly related to them?
    2. Fit a multiple linear regression model to the full data. Give the values of the estimated regression coefficients and test $H_0 : \beta_1 = 0$.
    3. Remove x_1 from the model. By examining the residual plots, comment on whether there is any reason to doubt the assumptions of the reduced model.

21. Suppose that $Y_i \sim$ Poisson(μ) for $i = 1, 2, \ldots, n$, all independent, and consider testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$, where μ_0 is known.

    1. Write down the restricted maximum likelihood estimate $\hat{\mu}_0$ of μ under H_0 and the maximum likelihood estimate $\hat{\mu}$.
    2. Obtain the generalised likelihood ratio.
    3. Use Wilks' theorem to find the critical region of a test with approximate significance level α for large n.

22. Consider the data 5, 1, 3, 5, 5, 4, 3, 2 which are assumed to be independent realizations of the Poisson distribution with expectation μ. We want to test $H_0 : \mu = \mu_0$ with $\mu_0 = 3$.

1. Obtain the numerical value of the generalised likelihood ratio $\Lambda(\mathbf{y})$ and discuss about the distribution of this statistic to perform the test $H_0 : \mu = \mu_0$.
2. Use Wilk's theorem to test $H_0 : \mu = \mu_0$, perform the test and write your conclusions.
3. Using the normal approximation to the data, perform the test $H_0 : \mu = \mu_0$ and compare with the earlier results.

23. Suppose for $i = 1, 2, \ldots, n$, we have independent $Y_i \sim \mathrm{Bin}(r_i, p)$, where r_i is known. Using data $y_1, \ldots, y_n$, consider testing $H_0 : p = p_0$ against $H_1 : p \neq p_0$, where p_0 is known.

    1. Write down the restricted maximum likelihood estimate $\hat{p}_0$ of p under H_0 and the maximum likelihood estimate $\hat{p}$.
    2. Obtain the generalised likelihood ratio for this test.
    3. Use Wilks' theorem to find the critical region of a test with approximate significance level α, for large n.
    4. The following (25,10), (15,6), (30,10) are data pairs (r_i, y_i) from acceptance sampling in textile industry. Apply your results to build the generalized likelihood ratio and use Wilks' theorem with $\alpha = 0.05$ to test $H_0 : p = 0.3$.

24. Suppose that $Y \sim \mathrm{Bin}(r, \pi)$, where r is known.

    1. Show that this distribution is a member of the exponential family.
    2. Explain why the distribution is in canonical form and write down the natural parameter.
    3. Use the general results for $E\{a(Y)\}$ and $\mathrm{Var}\{a(Y)\}$ to verify that $E(Y) = r\pi$ and $\mathrm{Var}(Y) = r\pi(1 - \pi)$.

25. Suppose that $Y \sim \mathrm{N}(\mu, \sigma^2)$, where σ^2 is known.

    1. Show that this distribution is a member of the exponential family.
    2. Explain why the distribution is in canonical form and write down the natural parameter.
    3. Use the general results for $E\{a(Y)\}$ and $\mathrm{Var}\{a(Y)\}$ to verify that $E(Y) = \mu$ and $\mathrm{Var}(Y) = \sigma^2$.

26. Consider a sequence of independent Bernoulli trials, where each trial has success probability p. The number of failures observed until we obtain r successes is a negative binomial random variable $X \sim \mathrm{NB}(r, p)$ with probability mass function $\Pr(X = x) = \binom{x+r-1}{x} p^r (1 - p)^x$.

    1. Show that this distribution is a member of the exponential family.
    2. Is the distribution in canonical form? Which is the natural parameter?
    3. Using exponential family results, show that $E(X) = r(1 - p)/p$.
    4. Consider data $x_1, x_2, \ldots, x_n$. Determine the maximum likelihood estimate $\hat{p}$. Is this estimator unbiased? Justify your answer.
    5. Compute the Fisher information number for estimating p.

27. Consider the random variable $Y \sim \mathrm{Ber}(p)$.

    1. Show that this distribution is a member of the exponential family.
    2. Determine if the distribution is in canonical form and write down the natural parameter.

3. Use the general results for $E\{a(Y)\}$ and $\text{Var}\{a(Y)\}$ to determine that $E(Y)$ and $\text{Var}(Y)$.

28. Repeat the calculations of Exercise 27 for the following distributions: a) binomial, b) geometric, c) exponential, d) gamma, e) lognormal and f) chi-squared.

29. Consider the mean $\mu = E(Y)$ and variance $\sigma^2 = V(Y)$ of the random variable $Y \sim \text{Ber}(p)$. Determine if the variance is a function of the mean and if so, give its explicit formula $\sigma^2 = f(\mu)$.

30. Repeat the calculations of Exercise 29 for the following distributions of the exponential family: a) binomial, b) geometric, c) negative binomial, d) Poisson, e) exponential, f) chi-squared, g) gamma and h) lognormal.

31. Suppose that $Y_i \sim \text{Bin}(r_i, \pi_i)$ for $i = 1, 2, \ldots, n$, all independent, where the r_i are known, $\log\{\pi_i/(1 - \pi_i)\} = \beta_0 + \beta_1 x_i$ and x_i is a known covariate.

    1. Find the Fisher information matrix.
    2. Obtain the asymptotic distributions of the maximum likelihood estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ of β_0 and β_1.
    3. State the approximate standard errors of $\hat{\beta}_0$ and $\hat{\beta}_1$.

32. Suppose that the continuous random variables $Y_1, \ldots, Y_n$ have distributions depending on the parameters $\theta_1, \ldots, \theta_p$ and that their ranges do not depend on the parameters. Let $L(\theta; \mathbf{y})$ and $l(\theta; \mathbf{y})$ denote the likelihood and log-likelihood of the parameter vector θ, respectively.

    1. Show that
    $$\frac{\partial l(\theta; \mathbf{y})}{\partial \theta_j} = \frac{1}{L(\theta; \mathbf{y})} \frac{\partial L(\theta; \mathbf{y})}{\partial \theta_j}.$$
    2. Prove that
    $$E\left\{\frac{\partial l(\theta; \mathbf{Y})}{\partial \theta_j}\right\} = 0.$$
    3. By differentiating the identity in part 1 with respect to θ_k, prove that
    $$E\left\{-\frac{\partial^2 l(\theta; \mathbf{Y})}{\partial \theta_j \partial \theta_k}\right\} = E\left\{\frac{\partial l(\theta; \mathbf{Y})}{\partial \theta_j} \frac{\partial l(\theta; \mathbf{Y})}{\partial \theta_k}\right\}.$$

33. Consider the data on beetles in Practical 3. answer the questions below. Fit the logistic, probit and extreme value models in `R`. Which of these provides the best description of the data? Present the results in a clear and concise table.

34. Suppose that $Y_i \sim \text{Poisson}(\mu_i)$ for $i = 1, 2, \ldots, n$, all independent, where $\log(\mu_i) = \beta_0 + \beta_1 x_i$ and x_i is a known covariate.

    1. Find the Fisher information matrix.
    2. Obtain the asymptotic distributions of the maximum likelihood estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ of β_0 and β_1.
    3. State the approximate standard errors of $\hat{\beta}_0$ and $\hat{\beta}_1$.

35. Suppose that $Y_i \sim \text{Bin}(r_i, \pi_i)$ for $i = 1, 2, \ldots, n$, all independent, where the r_i are known, $\log\{\pi_i/(1 - \pi_i)\} = \beta_0 + \beta_1 x_i$ and x_i is a known covariate.

1. Show that the maximum likelihood estimate of π_i in the maximal model is y_i/r_i.
2. Obtain the generalised likelihood ratio.
3. Use Wilks' theorem to find the critical region of a test with approximate significance level α for large n.

36. In lectures we have surveyed the logistic, probit and extreme value (complementary log-log) links which are used for the analysis of proportions (binomial data). In principle, a link for proportion data is any continuous function that transforms $(0,1) \to \mathbb{R}$. For example, the probit link $\Phi^{-1}(\cdot)$ is the inverse of the standard normal cumulative distribution function.

    1. Do some research about the link using the inverse cauchy distribution; write its explicit expression and show that it satisfies the transformation $(0,1) \to \mathbb{R}$.
    2. Plot the link transformation and compare with other links mentioned. Can you see some advantages or drawbacks of the Cauchy?
    3. Analyze the beetle data of Practical 3 using the link `cauchit`. Compare what you obtain with the earlier results. Does it improve over these? Write your comments.

37. A researcher wishes to know if consumption of caffeine improves performance on a memory test. There were 30 volunteers for each dose of caffeine (x), in milligrammes, and the number of volunteers who achieved a grade A in the memory test (y) is recorded. Below are the results.

x	0	50	100	150	200	250	300	350	400	450	500
y	10	13	17	15	10	5	4	3	3	1	0

    1. Fit a logistic regression model to the data. Give the values of the estimated regression coefficients and assess the goodness of fit of the model.
    2. Add x^2 to the model. Is there evidence that this model is an improvement over the two-parameter one?
    3. Obtain the fitted values of the new model. Plot both the proportions and the fitted values against the doses.

38. Suppose that $Y_i \sim \text{Poisson}(\mu_i)$ for $i = 1, 2, \ldots, n$, all independent, where $\log(\mu_i) = \beta_0 + \beta_1 x_i$ and x_i is a known covariate.

    1. Show that the maximum likelihood estimate of μ_i in the maximal model is y_i.
    2. Obtain the generalised likelihood ratio.
    3. Use Wilks' theorem to find the critical region of a test with approximate significance level α for large n.

39. Suppose that $Y_i \sim \text{N}(\beta x_i, \sigma^2)$ for $i = 1, 2, \ldots, n$, all independent, where x_i is a known covariate and σ is known. The fitted values are $\hat{\mu}_i = \hat{\beta} x_i$ and the variance of Y_i is $V(\hat{\mu}_i) = \sigma^2$, and we have $V(x) = \sigma^2$. .

    1. Write down the Pearson residual e_i^P.
    2. Find the transformation $A(x)$.
    3. Obtain the Anscombe residual e_i^A.

40. In an experiment designed to assess the potency of two test preparations of an insecticide relative to a standard, 60 aphids were placed on each of 12 cabbage plants. The three insecticides (w) were then applied in various doses (x), in milligrammes per litre of water, to each of four plants. The number of aphids still alive after three days (y) is determined and the results are as follows:

x	1.2	2.4	4.8	9.6	1.2	2.4	4.8	9.6	1.2	2.4	4.8	9.6
w	1	1	1	1	2	2	2	2	3	3	3	3
y	43	37	26	15	35	27	18	7	52	44	36	28

Analyse the data by fitting probit regression models in which the probit of the proportion of aphids killed by the insecticide is related to the logarithm of the dose.

1. Plot the proportions against the logarithms of the dose by insecticide. What are your conclusions?
2. By comparing the deviance for the model which allows a different intercept and slope for each insecticide with that for one in which the slopes are the same, test whether the regression lines are parallel.
3. Test whether there are differences between the insecticides.

41. Consider again the cloth data in Practical 6. Show that the estimate of the dispersion parameter ψ is $\hat{\psi} = 2.194$.

42. The following relationships can be described by generalized linear models. For each one, identify the response variable and the explanatory variables, select a probability distribution for the response (justifying your choice) and write down the linear component.

1. The effect of age, sex, height, mean daily food intake and mean daily energy expenditure on a person's weight.
2. The proportions of laboratory mice that became infected after exposure to bacteria when five different exposure levels are used and 20 mice are exposed at each level.
3. The relationship between the number of trips per week to the super- market for a household and the number of people in the household, the household income and the distance to the supermarket.

43. In a cross-sectional study of skin cancer, the site of the tumour and its histological type were recorded for 400 patients. The contingency table below shows the number of patients (y) with each combination of tumour type and site.

	Site			
Histological Type	Head and Neck	Trunk	Extremities	Total
Hutchinson's melanotic freckle	22	2	10	34
Superficial spreading melanoma	16	54	115	185
Nodular	19	33	73	125
Indeterminate	11	17	28	56
Total	68	106	226	400

The null hypothesis is that tumour type and site are independent.

1. Express the null hypothesis as a log-linear model, explaining your notation and any additional constraints.
2. Obtain the expected values under the null hypothesis. Compare these with the observed values.
3. Find the deviance and the value of Pearson's goodness-of-fit test statistic. What is your conclusion?

44. In a prospective study on a new treatment for pneumonia, patients were randomly allocated to two groups each of 40 patients. One group received the new treatment and the other the standard one, and the responses were the time taken to recover. The contingency table below shows the number of patients (y) with each combination of treatment and time taken to recover.

	Time to Recover			
	Short	Medium	Long	Total
Standard	6	15	19	40
New	10	21	9	40

The null hypothesis is that the time taken to recover is the same for each treatment group.

1. Express the null hypothesis as a log-linear model, explaining your notation and any additional constraints.
2. Obtain the expected values under the null hypothesis. Compare these with the observed values.
3. Find the deviance and the value of Pearson's goodness-of-fit test statistic. What is your conclusion?

45. Suppose that $T_i \sim \text{Exp}(\lambda_i)$ for $i = 1, 2, \ldots, n$, all independent. Consider model M1, in which all the λ_i are parameters; and model M2 for which $\lambda = \lambda_1 = \cdots = \lambda_n$. That is, M1 and M2 are the maximal and the null models, respectively.

1. Derive the corresponding maximum likelihood estimates for M1 and M2.
2. Using your estimates, determine the maximum value of the (log) likelihood in each case, i.e. $\hat{l}_{max} = \hat{l}_{M1} = l(\hat{\lambda}_1, \ldots, \hat{\lambda}_n; y)$ and $\hat{l}_{null} = \hat{l}_{M2} = l(\hat{\lambda}; y)$.
3. Derive an expression for the Fisher information matrix for the parameters in each of M1 and M2, and write formulæ for confidence intervals for parameters estimates in each case.
4. Repeat all the computations above for the case when $T_i \sim \text{Exp}(\theta_i)$. Your analysis will consider models 1 and 2. Compare which results and comment on the similarities when they appear. **Hint**. When using $\text{Exp}(\lambda_i)$ in the first part of this problem, you will use the density $f_T(t) = \lambda_i \exp(-\lambda_i t)$ for $t \geq 0$, while for $\text{Exp}(\theta_i)$, you'll use $f_T(t) = \theta_i^{-1} \exp(-t/\theta_i)$ for $t \geq 0$. In both cases the density is zero for negative t.

46. The following values are lifetimes of electronic components: 1.86, 12.96, 13.74, 8.57, 2.54. Using the results of Exercise 45 or otherwise,

1. Plot the likelihood function of the data under the model $T_i \sim \text{Exp}(\lambda)$. Do another plot of the likelihood using the model $T_i \sim \text{Exp}(\theta)$.
2. Give confidence intervals for each case. Use $\alpha = 0.05$.

47. Suppose that $T_i \sim \text{Exp}(\lambda_i)$ for $i = 1, 2, \ldots, n$, all independent, where $\lambda_i = \beta x_i$ and x_i is a known covariate.

    1. Write down the likelihood for the data $t_1, \ldots, t_n$.
    2. Show that the maximum likelihood estimator of β is $\hat{\beta} = n / \sum_{i=1}^{n} x_i T_i$.
    3. Find the Fisher information.

48. The break strength t_i in MPa was recorded for $n = 5$ industrial ceramic components. It is assumed that the distribution of strength is associated with porosity index x_i, which is a quantity controlled in the manufacturing process. The data is (1,6.798), (3,21.223), (3,1.873), (5,0.1), (5,0.398), which is given as pairs (x_i, t_i).

    1. Using the model of Exercise 47, compute the maximum likelihood estimate of β.
    2. Compute its observed Fisher information.
    3. Using your results and $\alpha = 0.05$, give a confidence interval for $\hat{\beta}$.

49. Suppose that the survival time $T > 0$ of a patient has a Weibull distribution with probability density function
$$f(t) = \alpha \lambda t^{\alpha - 1} \exp(-\lambda t^{\alpha}),$$
where $\alpha > 0$ and $\lambda > 0$.

    1. Show that the survivor function is $S(t) = \exp(-\lambda t^{\alpha})$.
    2. Obtain the hazard function.
    3. Explain how the hazard function behaves for different values of α.

50. Using `R`, plot the hazard function for the Weibull distribution considering the scale parameter $\lambda = 1$ and the the following cases for the shape parameter $\alpha = 0.25, 0.5, 1, 1.5, 3$.

51. Consider a set of censored life times observations (δ_i, t_i) for $i = 1, \ldots, n$. Here δ_i indicates censoring, i.e. if $\delta_i = 1$ we observed $T_i = t_i$ and if $\delta_i = 0$ then we had $T_i > t_i$. Assume that the times t_i follow an exponential distribution.

    1. Using the null model, derive an expression for the mle of λ.
    2. Compute the Fisher information number and by plugging-in the mle $\hat{\lambda}$, give a formula for the observed Fisher information.
    3. Using the earlier results give a formula for the estimated standard error of $\hat{\lambda}$ and for a $100(1 - \alpha)$ confidence interval for $\hat{\lambda}$.

52. Consider the data (1,0.62), (0,4.32), (1,4.58), (1,2.86), (1,0.85), (0,5.28), (1,0.1), (1,2.27), (1,21.22), (0,1.87), where the pairs are (δ_i, t_i) as above. Using the results of Exercise 51, estimate λ, compute its estimated variance and give a confidence interval for $\hat{\lambda}$ using $\alpha = 0.05$.

53. Consider the data (1,63.67), (0,5.62), (1,0.5), (1,1.99), (1,10.09), (0,13.44), (1,41.13), (1,28.24), (0,39.36), (1,17.59), (1,15.98), (0,13.05), where the pairs are (δ_i, t_i) as above. Using the results of Exercise 51, estimate λ, compute its estimated variance and give a confidence interval for $\hat{\lambda}$ using $\alpha = 0.05$.