Week 12 Mon.

goal: to prove determinants are multiplicative i.e.

$\det(AB) = \det(A) \cdot \det(B)$ for any two square matrix of the same size

Theorem 8.3.11. If A is an $n \times n$ matrix and E is an elementary $n \times n$ matrix then

$$\det(EA) = \det(E) \cdot \det(A)$$

with

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ is type I (interchanging two rows)} \\ \alpha & \text{if } E \text{ is type II (multiplying a row with } \alpha) \\ 1 & \text{if } E \text{ is type III (adding a multiple of one row to another)} \end{cases}$$

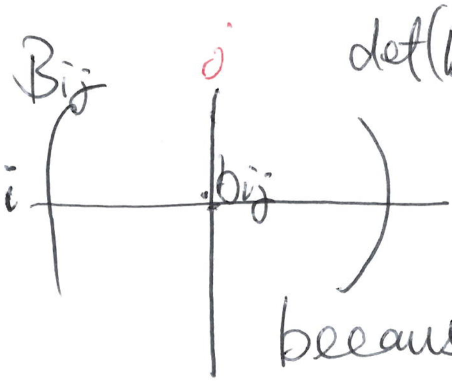
recall. an elementary matrix differs from the identity matrix by one simple elementary row operation.

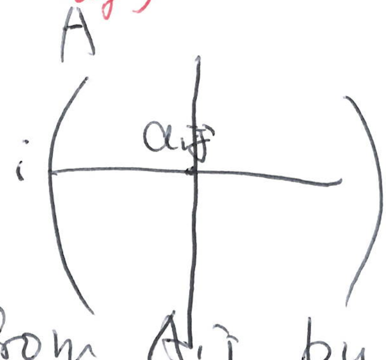
Proof: By induction on the size of A, which is n.

- If $n = 2$. A is an 2×2 matrix. following theorem 8.1.2. We have $\det(EA) = \det(E) \cdot \det(A)$
- Suppose this theorem holds for $n = k$, where $k \geq 2$.
- (We want to check the theorem holds for $n = k+1$ as well)

- Let A be $(k+1)\times(k+1)$ matrix. we write $\underset{n\times n}{B} = \underset{n\times n}{E}\,\underset{n\times n}{A}$. Expand $\det(EA)$ across a row, which is unaffected by the ~~the~~ action of E on A. say, row i. We have $\Rightarrow (b_{ij} = a_{ij})$ ①

$$\det(B_{ij}) = r.\det(A_{ij}) \quad ②$$





because B_{ij} is obtained from A_{ij} by the same type of elementary row operation the E performs on A, where

$r = -1, \alpha, 1$. depending on the type of E

b_{ij} but $b_{ij} = a_{ij}$ for row i

$$\det(EA) = \det(B) = \sum_{j=1}^{k+1} a_{ij}(-1)^{(i+j)}\det(B_{ij})$$

$$\det(B) = \sum_{j=1}^{k+1} b_{ij}(-1)^{i+j}\det(B_{ij})$$
$$= \sum_{j=1}^{k+1} a_{ij}(-1)^{i+j}\det(B_{ij})$$

$$= \sum_{j=1}^{k+1} a_{ij}(-1)^{(i+j)}\, r.\det(A_{ij})$$

$$= r\sum_{j=1}^{k+1} a_{ij}(-1)^{(i+j)}\det(A_{ij})$$

$$= r.\det(A)$$

$$= \det(E).\det(A)$$

Summary:

Thus. we can see the theorem is true for 2×2 matrices. and the truth of the theorem for $k\times k$ matrices $(k\geq 2)$ implies the truth of the theorem for $(k+1)\times(k+1)$ matrice. By the Principle of Induction, the theorem holds for square matrice of any size

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Theorem 8.3.12 If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\cdot\det(B)$$

Proof: Case I: If A is not invertible, then neither is AB. (for otherwise

$$A(B(AB)^{-1}) = A(B\cdot B^{-1}\cdot A^{-1}) = A(B\cdot B^{-1})A^{-1}$$

$$= A\cdot I\cdot A^{-1} = A\cdot A^{-1} = I \quad)$$

Thus by theorem 8.3.5

$$\det(AB) = 0 = 0\cdot \det(B) = \det(A)\cdot\det(B)$$

Case II If A is invertible then the invertible Matrix theorem, that is there exist elemantry matrices $E_1, \dots E_k$ such that

$$A = E_k E_{k-1} \dots E_1$$

$$\det(A) = |A|$$

$$|AB| = |E_k E_{k-1} \dots E_1 \cdot B| = |E_k||E_{k-1}E_{k-2}\dots E_1 B|$$

$$= |E_k||E_{k-1}||E_{k-2}|\dots|E_1||B|$$

$$= |E_k E_{k-1}||E_{k-2}|\dots|E_1||B|$$

$$= |E_k E_{k-1} E_{k-2} \dots E_1||B|$$

$$= |A|\cdot|B|$$

Corollary 8.3.13. If A is an invertible matrix then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

proof. Since A is invertible, we have

$$A^{-1} \cdot A = I.$$

$$\det(A^{-1} \cdot A) = \det(I) = 1$$

Thus base on the previous theorem (8.3.12).

$$\det(A^{-1} \cdot A) = \det(A^{-1}) \cdot \det(A) = 1$$

We know $\det(A) \neq 0$, because A is inveritible.

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

(Theorem 8.3.5)

8.4 Cramer's rule.

We will use determinants to solve system of linear equation $\underset{n\times n}{A}x = b$ for the case of quadratic and invertible $n \times n$ matrices.

We recall that the system has a unique solution $\underset{n\times 1}{x} = \underset{n\times n}{A^{-1}}\, \underset{n\times 1}{b}$

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}}$$

- We will show how to derive the Cramer's rule.

We first introduce the matrices

$$B_i := (A^1, A^2, \dots, A^{i-1}, b, A^{i+1}, \dots A^n),$$

which are obtained from the matrix A by replacing the ith column with b.

We can write down the linear system

$$Ax = b \quad \text{as}$$

$$\sum_j x_j \cdot A^j = b, \quad \text{where } A^j \text{ is the jth}$$

column of matrix A.

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\begin{array}{ccccccc} & A^1 & & A^2 & & & A^n \\ x_1 & \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix} & + \; x_2 & \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{pmatrix} & + \dots + & x_n & \begin{pmatrix} a_{n1} \\ \vdots \\ a_{nn} \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

By applying the properties of determinants.

$$\det(B_i) = \det(A^1, A^2, \dots, A^{i-1}, b, A^{i+1}, \dots A^n)$$

additive property of determinants

$$= \det\Big(A^1, A^2, \dots, A^{i-1}, \sum_j x_j \cdot A^j, A^{i+1}, \dots A^n\Big)$$

if A, B, C are identical except at a fixed row/column i, $j \in [1, n]$, if that row/column A^i equals to $B^i + C^i$, $\det(A) = \det(B) + \det(C)$

ith column

$$= \det(A^1, A^2, \dots A^{i-1}, x_1 A^1, A^{i+1}, \dots A^n)$$
$$+ \det(A^1, A^2, \dots A^{i-1}, x_2 A^2, A^{i+1}, \dots A^n)$$
$$+ \dots$$
$$+ \det(A^1, A^2, \dots A^{i-1}, x_n A^n, A^{i+1}, \dots A^n)$$

$$\Big(A^1, A^2, \dots \sum_j x_j A^j, A^{i+1}, \dots A^n\Big)$$
$$= (A^1, A^2, \dots A^{i-1}, x_1 A^1, A^{i+1}, \dots A^n)$$
$$+ (A^1, A^2, \dots A^{i-1}, x_2 A^2, A^{i+1}, \dots A^n)$$
$$+ \dots$$
$$+ (A^1, A^2, \dots A^{i-1}, x_n A^n, A^{i+1}, \dots A^n)$$

$$= x_1 \underbrace{\det(A^1, A^2, \dots A^{i-1}, A^1, A^{i+1}, \dots A^n)}_{0}$$
$$+ x_2 \underbrace{\det(A^1, A^2, \dots A^{i-1}, A^2, A^{i+1}, \dots A^n)}_{0}$$
$$+ \dots$$

Except the ith item

$$+ x_i \det(A^1, A^2, \dots A^{i-1}, A^i, A^{i+1}, \dots A^n)$$
$$\vdots$$
$$+ x_n \underbrace{\det(A^1, A^2, \dots A^{i-1}, A^n, A^{i+1}, \dots A^n)}_{0}$$

Recall, the fact that the determinant of a matrix with two identical columns is equal to 0

$$= x_i \det(A^1, A^2, \dots A^{i-1}, A^i, A^{i+1}, \dots A^n)$$
$$= x_i \det(A)$$

Thus
$$x_i = \frac{\det(B_i)}{\det(A)} = \frac{\det(A^1, A^2, \dots A^{i-1}, b, A^{i+1}, \dots A^n)}{\det(A)}$$

Example 8.4.1 Let us apply Cramer's rule to a linear syse $Ax = b$ with

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -2 \\ 0 & 3 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} 2x_1 - x_2 &= 1 \\ x_1 + 2x_2 - 2x_3 &= 2 \\ 3x_2 + 4x_3 &= 0 \end{aligned}}$$

By replace the corresponding calum of A withe the vector b, We have three matrices corresponding to the three variable x_1, x_2, x_3.

$$B_1 = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & -2 \\ 0 & 3 & 4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & 0 & 4 \end{pmatrix} \quad B_3 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 0 & 3 & 0 \end{pmatrix}$$

$\det(B_1) = 22$, $\det(B_2) = 12$, $\det(B_3) = -9$

$\det(A) = 32$, thus

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{22}{32}$$

$$x_2 = \frac{\det(B_2)}{\det(A)} = \frac{12}{32}$$

$$x_3 = \frac{\det(B_3)}{\det(A)} = -\frac{9}{32}$$

$$x = \frac{1}{32} \begin{pmatrix} 22 \\ 12 \\ -9 \end{pmatrix}$$