Week 11 Fri

Theorem 8.3.5. A matrix A is invertible <u>if and only if</u> $\det(A) \neq 0$.

Proof: Bring A to row echelon form U (which is necessarily upper triangular) using elementary row operations. In the process we only ever multiply a row by a non-zero scalar. according to Theorem 8.3.1 we have

$\det(\overset{A}{\cancel{U}}) = r \det(\overset{U}{\cancel{A}})$ for some $r \neq 0$. If

- A is invertible, then $\det(U) = 1$ by Theorem 8.2.8 as the entries in the row echelon form U are 1s (the leading 1s).

  Thus $\det(A) = r \det(U) = r \neq 0$.

- if A is not invertible, the at least one diagonal entry of U is zero. thus $\det(U) = 0$.

  $\det(A) = r \cdot \det(U) = 0$.

Definition 8.3.6 A square matrix A is called ~~Sigu~~ singular if $\det(A) = 0$. otherwise it is called to be nonsigular.

Corollary 8.3.7. A matrix is invertible if and only if it is non singular.

Theorem 8.3.8. If A is an $n \times n$ matrix, the $\det(A) = \det(A^T)$

- First, check the example for any 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

check $\det(A) = \det(A^T)$ by

doing the cofactor expansion of $\det(A)$ across the first row and of $\det(A^T)$ across the first column

$$\det(A) = a_{11} \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{B} + (-1)^3 a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{2} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\det(A^T) = a_{11} \underbrace{\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix}}_{B^T} + (-1)^{2+1} a_{12} \begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{3} \\ a_{23} & a_{33} \end{vmatrix}$$

$$\det(B) = a_{22} \cdot a_{33} - a_{32} \cdot a_{23}$$
$$\det(B^T) = a_{22} a_{33} - a_{23} \cdot a_{32}$$

We can see for a 3×3 matrix A, $\det(A) = \det(A^T)$

proof. The proof is by induction on n. which is the size of the square matrix A.

when $n=1$ $A=(a_{11})$ $A^T=(a_{11})$

$\det(A)=a_{11}=\det(A^T)$

$n=2$ $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $A^T=\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$

$\det(A)=\det(A^T)$

$\vdots$

- Suppose now it has been proved for $k\times k$ matrices for some interger k. now we want to show the theorem is true for $(k+1)\times(k+1)$ matrix

- Let A be a $(k+1)\times(k+1)$ matrix. the (i, j) cofactor of A equals the (i, j) cofactor of A^T, because the cofactor involves $k\times k$ determants only. determinant only

Thus the theorem holds for a $(k+1)\times(k+1)$ matrix if it holds for $k\times k$ matrixe

Hence, Cofactor expansion of det(A) across frist row = cofactor expansion of $\det(A^T)$ across the first colum.

Theorem 8.3.9. Let A be a square matrix.

a) If two columns of A are interchanged to produce B. ~~ote~~ then. $\det(B) = -\det(A)$

b) If one colum of A is multipled by α to produce B. then $\det(B) = \alpha \cdot \det(A)$

c) If a multiple of one column of A is added to another colum to produce matrix B. then $\det(B) = \det(A)$

Example 8.3.10.

Find $\det(A)$, where $A = \begin{pmatrix} 1 & 3 & 4 & 8 \\ -1 & 2 & 1 & 9 \\ 2 & 5 & 7 & 0 \\ 3 & -4 & -1 & 5 \end{pmatrix}$

Sol: $\det(A) = \begin{vmatrix} 1 & 3 & 4 & 8 \\ -1 & 2 & 1 & 9 \\ 2 & 5 & 7 & 0 \\ 3 & -4 & -1 & 5 \end{vmatrix} \underset{C_1+C_2 \to C_2}{=} \begin{vmatrix} 1 & 4 & 4 & 8 \\ -1 & 1 & 1 & 9 \\ 2 & 7 & 7 & 0 \\ 3 & -1 & -1 & 5 \end{vmatrix}$

$C_2 - C_3 \to C_2$

$= \begin{vmatrix} 1 & 0 & 4 & 8 \\ -1 & 0 & 1 & 9 \\ 2 & 0 & 7 & 0 \\ 3 & 0 & 1 & 5 \end{vmatrix} = 0$

If we have two colums / two rows equal in a matrix. its determinent is 0.