Partial Summary of ~~Week 10~~ 8-9

- Elementary matrices are matrices which differs from the identity matrix by one single elementary row operation

Type I: Row switching , $R_i \leftrightarrow R_j$

Type II: Row multiplication , $kR_i \to R_i$

Type III: Row addition ,

$R_i + kR_j \to R_i$

- If E is an elementary matrix, the E is invertible and E^{-1} is an elemetary matrix of the same type. (Theorem 7.6.6)

- Invertible Matrix Theorem (Theorem 7.6.9) $\longrightarrow$ a simple method inverting matrices

a) A is invertible
b) $AX=0$ has ~~the~~ only the trivial solution
c) A is row equivalent to I
d) A is a product of elementary matrices

- Thus,

A is invertible

$\Updownarrow$

$E_k \cdot E_{k-1} \cdots E_1 A = I$

$\Updownarrow$

$E_k \cdot E_{k-1} \cdots E_1 \cdot A \cdot A^{-1} = I \cdot A^{-1}$,

$E_k \cdot E_{k-1} \cdots E_1 = A^{-1}$

Week 10. Continue with definition 8.1.3

If $A = (a_{ij})$ is a 3×3 matrix, its determinant $\det(A)$ is defined by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\det(A_{11})} - a_{21}\underbrace{\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}}_{\det(A_{21})} + a_{31}\underbrace{\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}_{\det(A_{31})}$$

Note the determinant of a 3×3 matrix A is given in terms of determinants of certain 2×2 submatrices of A.

8.2. General definition of determinants.

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the i-th row and jth column of A

Example 8.2.1 If $A = \begin{pmatrix} 3 & 2 & 5 & -1 \\ 2 & 9 & 0 & 6 \\ 7 & -2 & 3 & 1 \\ 4 & -5 & 8 & -4 \end{pmatrix}$ a_{23}

$$A_{23} = \begin{pmatrix} 3 & 2 & -1 \\ 7 & -2 & 1 \\ 4 & -5 & -4 \end{pmatrix}$$

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Definition 8.2.2. Let $A = (a_{ij})$ be an $n \times n$ matrix. Then $\det(A)$ is defined as

- If $n = 1$ then $\det(A) = a_{11}$
- if $n > 1$ then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det(A_{i1})$$

$$= a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + a_{31}\det(A_{31}) \cdots + (-1)^{n+1} a_{n1} \det(A_{n1})$$

$\boxed{\text{Using the first column of } A.\ a_{i1},\ i \in [1, n]}$

Example 8.2.3. Compute the determinant of

$$A = \begin{pmatrix} 0 & 0 & 7 & -5 \\ -2 & 9 & 6 & -8 \\ 0 & 0 & -3 & 2 \\ 0 & 3 & -1 & 4 \end{pmatrix}$$

Sol:

$$\det(A) = \begin{vmatrix} 0 & 0 & 7 & -5 \\ -2 & 9 & 6 & -8 \\ 0 & 0 & -3 & 2 \\ 0 & 3 & -1 & 4 \end{vmatrix} = -(-2) \begin{vmatrix} 0 & 7 & -5 \\ 0 & -3 & 2 \\ 3 & -1 & 4 \end{vmatrix}$$

$$= 2 \cdot 3 \begin{vmatrix} 7 & -5 \\ -3 & 2 \end{vmatrix}$$

$$= 2 \cdot 3 \, (7 \cdot 2 - (-3) \cdot (-5))$$

$$= 2 \cdot 3 \cdot (-1) = -6.$$

Definition 8.2.4. Given a square matrix $A = (a_{ij})$ the (i,j)-cofactor of A is denoted as C_{ij}

number

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Theorem 8.2.5. (Cofactor Expansion Theorem). The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any column or row of A.

The expansion down the j-th column is

Fix j, vary the index of rows

$$\det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

and the expansion across the i-th row is

Fix i vary the index of colum

$$\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

Thus. our previous definition in 8.2.2. is the cofactor expansion down the 1-th column

$$\det(A) = a_{11} C_{11} + a_{21} C_{21} + \cdots + a_{n1} C_{n1}$$

The factor $(-1)^{i+j}$ has the pattern of sign

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{pmatrix}$$

Example 8.2.6. Use a cofactor expansion across a given row or column to compute det A. where

$$A = \begin{pmatrix} 4 & -1 & 3 \\ 0 & 0 & 2 \\ 1 & 0 & 7 \end{pmatrix}$$

Solution: (We can use the colum or row which has the most number of 0s)

Here, we can use the 2nd row or column

Using a cofactor expansion across the second row we have

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= (-1)^{2+1}\underbrace{a_{21}}_{=0}\det(A_{21}) + (-1)^{2+2}\underbrace{a_{22}}_{=0}\det(A_{22}) + (-1)^{2+3}a_{23}\det(A_{23})$$

$$= -2\begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix} = -2(4\cdot 0 - (-1)\cdot 1)$$

$$= -2$$

Example 8.2.7. Compute $\det(A)$, where

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ -2 & 5 & 0 & 0 & 0 \\ 9 & -6 & 4 & -1 & 3 \\ 2 & 4 & 0 & 0 & 2 \\ 8 & 3 & 1 & 0 & 7 \end{pmatrix}$$

Solution: (We can see the cofactor expansion across the 1st row or the 4th colum will be most effecient in computing the determinat)

$$\det A = 3 \begin{vmatrix} 5 & 0 & 0 & 0 \\ -6 & 4 & -1 & 3 \\ 4 & 0 & 0 & 2 \\ 3 & 1 & 0 & 7 \end{vmatrix}$$

$$= 3 \cdot 5 \cdot \begin{vmatrix} 4 & -1 & 3 \\ 0 & 0 & 2 \\ 1 & 0 & 7 \end{vmatrix}$$

(now we do the cofactor expansion across the 2nd row of the submatrix)

$$= 3 \cdot 5 \cdot 2 (-1)^{2+3} \begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= -30 \left(4 \cdot 0 - (-1) \cdot 1 \right)$$

$$= -30$$

Theorem 8.2.8 If A is either an upper or a lower triangular matrix. then $\det(A)$ is the product of the diagnol entries of A

8.3. Properties of determinants

Theorem 8.3.1. Let A be a square matrix.

a) If two rows of A are interchanged to produce B. then $\det(B) = -\det(A)$

b) If one row of A is multiplied by α. (scalar) to produce B. then $\det(B) = \alpha . \det(A)$

c) If a multiple of one row of A is added to another row to reproduce a matrix B. then $\det(B) = \det(A)$

Example 8.3.2

a) $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = - \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{vmatrix}$$

b) $$\underbrace{\begin{vmatrix} 0 & 1 & 2 \\ 3 & 12 & 9 \\ 1 & 2 & 1 \end{vmatrix}}_{B} = 3 \underbrace{\begin{vmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 1 & 2 & 1 \end{vmatrix}}_{A}$$

c) $\begin{vmatrix} 3 & 1 & 0 \\ 4 & 2 & 9 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 7 & 3 & 9 \\ 0 & -2 & 1 \end{vmatrix}$

$\qquad\qquad A \qquad\qquad\qquad\qquad B$

Homework:
1. compute the determinants of LHS. RHS and check whether it is true
2. Find out which row operations are down to get B from A

Show How to use the Cofactor expansion and the Theorem 8.3.1 to compute the determinats of the following matrices.

Example 8.3.3. Compute. det(A), where

$$A = \begin{pmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{pmatrix}$$

Sol:

$$\det(A) \underset{R_3+2R_1 \to R_3}{=} \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & 6 \\ -5 & -8 & 0 & 9 \end{vmatrix} \underset{R_2-R_3 \to R_3}{=} \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 0 & 0 & 0 \\ -5 & 8 & 0 & 9 \end{vmatrix}$$

$= 0$

Example 8.3.4

Compute $\det(A)$, where

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{pmatrix}$$

Solution:

$$\det A \underset{\substack{R_2+R_4 \\ \to R_4}}{=} \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

Theorem 8.3.1 c)

$$= 2 \cdot (-1)^{2+1} \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$

Cofactor expansion across the 2nd row of the matrix in the last step

$$= -2 \underset{\substack{R_2-3\cdot R_1 \\ \to R_2}}{} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

Theore 8.3.1 c)

$$= -2 \cdot 5 \cdot (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix}$$

Cofactor expansion across the 2nd row again

$$= 10\,(1 \cdot (-3) - 0 \cdot 2)$$

$$= -30$$