Office Hour in learning Cafe

Thursdays 14:00 – 15:00

MB - B11 School's social Hub

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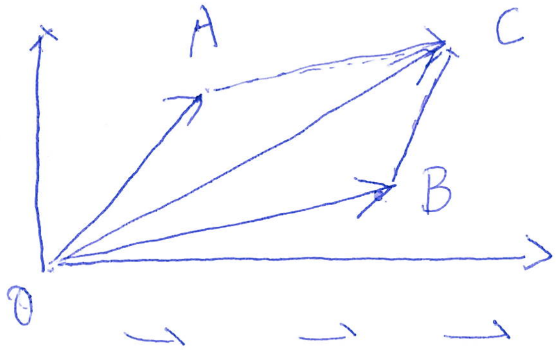
- jonit hour lecture Week 2

continue from 3.1.4



$$\vec{OC} = \vec{OA} + \vec{OB}$$

Corollary 3.1.5. Let OBCA be the parallelogram above. Then.

$$\vec{OC} = \vec{OA} + \vec{AC}$$
$$\vec{OC} = \vec{OB} + \vec{BC}$$

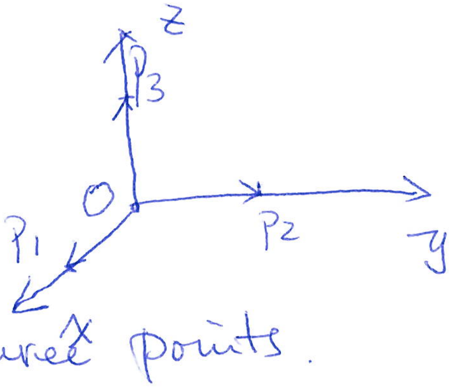
$\Rightarrow$ Page 20 Position Vectors in R^3

the origin point. $O = (0, 0, 0)$,

$P_1 = (1, 0, 0)$

$P_2 = (0, 1, 0)$

$P_3 = (0, 0, 1)$



connecting the origin with three points.

P_1, P_2, P_3

$\vec{OP_1} = \mathbf{i}$

$\vec{OP_2} = \mathbf{j}$

$\vec{OP_3} = \mathbf{k}$

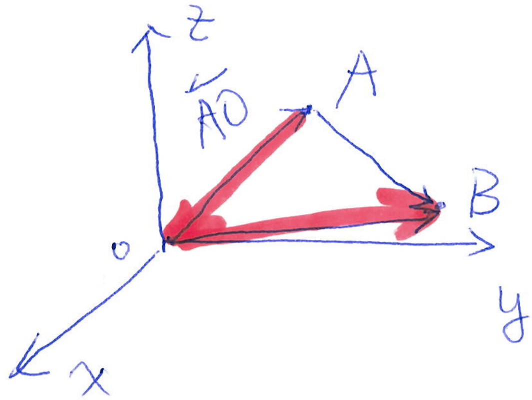
position vectors.

$\Rightarrow$ back to Page 22

Definition of the vectors $\overrightarrow{AB}$ in R^3 and express it in terms of $\mathbf{i}$. $\mathbf{j}$. $\mathbf{k}$.

proposition 3.1.6. Let $A = (x_A, y_A, z_A)$, $B = (x_B, y_B, z_B)$ be two points in R^3. The vectors $\overrightarrow{AB}$ is the sum

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$



$$\overrightarrow{AO} = \begin{pmatrix} -x_A \\ -y_A \\ -z_A \end{pmatrix} \quad \overrightarrow{OB} = \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix}$$

$$\overrightarrow{AB} = \underline{\underline{(x_B - x_A)}}\,\mathbf{i} + \underline{\underline{(y_B - y_A)}}\,\mathbf{j} + \underline{\underline{(z_B - z_A)}}\,\mathbf{k}$$

$$\overrightarrow{AO} = \underbrace{-1}_{\text{Scalar}} \cdot \overrightarrow{OA}$$

Conclusion: every vector in $\mathbb{R}^3$ can be regarded as a tripple of real numbers. Eg: $\overrightarrow{OA} = \begin{pmatrix} x_A, \\ y_A, \\ z_A \end{pmatrix}$.

$$\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$$

To distiguish vectors from points, we use columns for vectors, rows for points.

3.2 Vectors in $\mathbb{R}^n$

In the Sequel $\mathbb{R}^n$ is the set of Colum Vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

of real numbers. Addition and multiplication by real scalars are ~~as~~ defined as in

Definition. 3.1.3.

for

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ w_n + w_n \end{pmatrix}$$

for any $\lambda \in \mathbb{R}$

$$\lambda \mathbf{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

We recall addition and multiplication in $\mathbb{R}$ fulfill the following properties:

i) addition in $\mathbb{R}$ is commutative

for all $x, y, \in \mathbb{R}$ $\quad x+y=y+x$

ii) ------- is associative

for all $x, y, z \in \mathbb{R}$. $\quad (x+y)+z = x+(y+z)$

iii) 0 is the identity for the addition

for all $x \in \mathbb{R}$ $\quad x+0 = x$

iv) $-x$ is the additive inverse of x

for all $x \in \mathbb{R}$. $\quad x+(-x)=0$

v) multiplication in $\mathbb{R}$ is also assoiative

for all $x, y, z \in \mathbb{R}$

$$(x \cdot y) z = x \cdot (y \cdot z)$$

vi) multipliation in $\mathbb{R}$ is commutative.

for all $x, y, \in \mathbb{R}$

$$xy = yx$$

vii) identity for the multipliaction is 1

for all $x \in \mathbb{R}$, $1 \cdot x = x$

viii) The multipliation inverse of x is $\frac{1}{x}$

for all $x \neq 0 \in \mathbb{R}$ $\quad x \cdot \left(\frac{1}{x}\right) = 1$

ix) for all $x, y, z \in \mathbb{R}$ $\quad x(y+z) = x \cdot y + x \cdot z$

(distributive property)

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These propertises are inherited by the operations define in $\mathbb{R}^n$ as the operations are defined componentwise.

$\boxed{\text{Proposition 3.2.1}}$ $\mathbb{R}^n$ is a set closed with respect to addition and scalar multiplication. In addition, the following properties holds

i) for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

ii) for all $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$

$$(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z})$$

iii) for all $\mathbf{v} \in \mathbb{R}^n$ (identity for addition)

$$\mathbf{v} + \mathbf{0} = \mathbf{v} \qquad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(iv) $-\mathbf{v}$ is the additive inverse of $\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0} \qquad \vec{0}$$

v) multiplication by scalars is associative for all $\mathbf{v} \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}$

$$(\alpha \cdot \beta)\mathbf{v} = \alpha(\beta \cdot \mathbf{v})$$

vi) the identity for the multiplication by scalars is 1
for all $\mathbf{v} \in \mathbb{R}^n$.

$$1\mathbf{v} = \mathbf{v}$$

vii) distributive property

for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$,

$$\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$$

viii) distributive property

for all $\mathbf{v} \in \mathbb{R}^n$. $\alpha, \beta \in \mathbb{R}$

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$$

Definition 3.2.2. A set V with ~~and~~ two operations (addition and multiplication by scalars) fulfilling the properties of the proposition above (3.2.1) is called vector space.

if scalars are real, we mean
real vector spaces

if scalar are complex $\mathbb{C}$,
complex vector spaces.

Now we introduce following notion of Standard Vectors. See how vectors in $\mathbb{R}^n$ can be expressed in terms of them

Definition 3.2.3. Let $i = 1, 2, \ldots n$, The standard vector e_i is the column vector with ith entry equals to 1 and all the others equal to 0.

we therefore have

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

It is common to use the notation $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ in R^2

$\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e} = \mathbf{j}$, $\mathbf{e} = \mathbf{k}$ in R^3

Proposition 3.2.4. Every vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written as a unique linear combination of standard vectors, i.e. there exists a unique choice of $v_i \in R$, $i = 1, \dots, n$ such that

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$$

Proof. We need to prove existence and uniqueness of the components v_i for $i = 1, \dots, n$

by the definition of the sum of vectors and multiplication by scalars, we have

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + v_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$= v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \dots + v_n \mathbf{e}_n$$

$$= \sum_{i=1}^{n} v_i \mathbf{e}_i$$

This proves the existence part.

Now the ~~proo~~ proof of uniqueness.

We assume $\mathbf{v}$ can be written as

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{pmatrix}$$

It follow

$$\mathbf{v} - \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} - \begin{pmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This leads to $v_i = v_i'$

Example 3.2.5

Let $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{w} = -4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

Hence by the properties of the addition of vectors and multiplication by scalars we have

$$\mathbf{v} + \mathbf{w} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} - 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

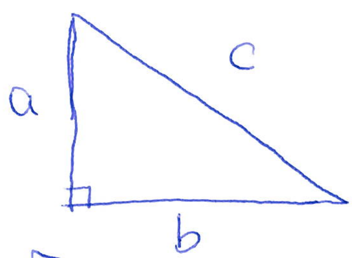
$$= (1-4)\mathbf{i} + (-2+1)\mathbf{j} + (5-3)\mathbf{k}$$

$$= -3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

Definition 3.2.6. Let $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$ be a vector in R^n. Its length (or norm or module) is defined as

$$|\mathbf{v}| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

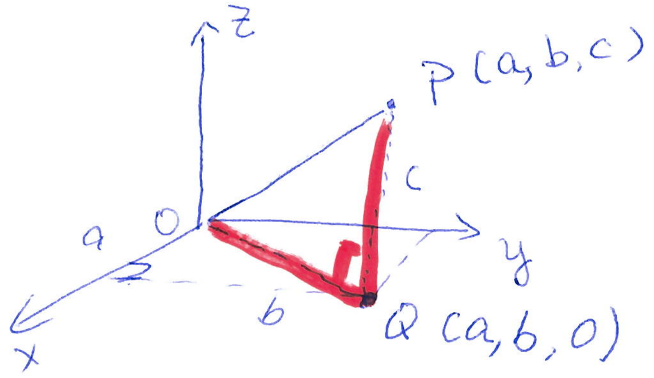
Pythagoras Theorem.



$c = \sqrt{a^2 + b^2}$

by multiplying $\mathbf{v}$ by the scalar $\frac{1}{|\mathbf{v}|}$, we obtain a unit vector, i.e. a vector with norm 1.

Proposition 3.2.7. Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Let P be the point in $\mathbb{R}^3$ with coordinates (a, b, c). Then $|\mathbf{u}|$ is the length of the segment OP.



proof: We assume $P \neq O$, otherwise the statement is trivial. We project point P on the xy-plane and get the point $Q = (a, b, 0)$.

Pythagoras Theorem

The length of the segment $OQ = \sqrt{a^2+b^2}$,

The we can consider the segment OP

$$= \sqrt{OQ^2 + c^2} = \sqrt{a^2+b^2+c^2}$$

$$|\mathbf{v}| = \sqrt{a^2+b^2+c^2} \quad \text{(Definition 3.2.6)}$$

Conclusion of chapter 3

Vectors R^2 R^3 R^n

Vector space

standard vectors

length of vectors