Actuarial Mathematics II
MTH5125

Joint Life and Last Survivor Benefits

Dr. Melania Nica

Spring Term

Plan for two weeks

- Based on Chapter 10, DHW
- Joint distribution of future life
- Joint life status
- Last survivor status
- Force of mortality for joint life/last survivor
- Life tables
- Curtate joint life/last survivor future lifetime
- Life Insurance benefits for joint life/last survivor
- Annuities for joint life/last survivor policies
- Contingent functions and Reversionary annuities
- Multiple state framework

Introduction

- The theory that we have developed for the death benefit of a single life can be extended to multiple lives (in this class, two lives (x) and (y)).
- Unless otherwise stated, we will assume that the two future lifetime random variables are independent.
- The survival of the two lives is referred to as the *status of interest* or simply the *status*.

Introduction

- There are two common types of status:
  - The joint-life status is one that requires the survival of both lives.
    - the status terminates on the first death of one of the two lives.
  - The last-survivor status is one that ends upon the death of both lives.
    - the status survives as long as at least one of the component members remains alive.

Introduction

The most common types of benefit which are contingent on two lives are:

Annuities:

- joint life annuity - payable until the first death of (x) and (y)
- last survival annuity - payable until the second death of (x) and (y)
- reversionary annuity: starts on the first death ends on the second death

Life insurance:

- Joint life insurance: pays benefit on the first death (x) and (y)
- A last survivor insurance: pays benefit on the second death (x) and (y)
- Contingent insurance pays benefit contingent on multiple events: for example benefit paid on tthe death of (y) if (x) is still alive

Joint distribution of future lifetimes

Consider the case of two lives currently aged (x) and (y) with respective with future lifetimes T_x and T_y.

- Joint cumulative distribution function:

$$F_{T_x T_y}(s, t) = P(T_x \leq s, T_y \leq t)$$

  - Independence:

$$\begin{aligned} F_{T_x T_y}(s, t) &= P(T_x \leq s, T_y \leq t) \\ &= P(T_x \leq s) \times P(T_y \leq t) \\ &= F_x(s) \times F_y(t) \end{aligned}$$

Joint distribution of future lifetimes

- Joint density function:

$$f_{T_x T_y}(s,t) = \frac{\partial^2 F_{T_x T_y}(s,t)}{\partial s \partial t}$$

  - Independence

$$f_{T_x T_y}(s,t) = f_x(s) \times f_t(t)$$

Joint distribution of future lifetimes

- Joint survival distribution function:

$$S_{T_x T_y}(s,t) = P(T_x > s, T_y > t)$$

  - Independence:

$$\begin{aligned} S_{T_x T_y}(s,t) &= P(T_x > s, T_y > t) \\ &= P(T_x > s) \times P(T_y > t) \\ &= S_x(s) \times S_y(t) \end{aligned}$$

The joint life status

- Status that survives so long as all members are alive, and therefore fails upon the first death.
- Notation: (xy) for two lives (x) and (y)
- For two lives: $T_{xy} = \min(T_x, T_y)$

The joint life status distribution

Cumulative distribution function:

$$\begin{aligned} F_{T_{xy}}(t) &= {}_tq_{xy} = P\left[\min\left(T_x, T_y\right) \le t\right] \\ &= 1 - P\left[\min\left(T_x, T_y\right) > t\right] \\ &= 1 - P\left[T_x > t, T_y > t\right] \\ &= 1 - S_{xy}(t) \\ &= 1 - {}_tp_{xy} \end{aligned}$$

${}_tp_{xy}$ - the probability that both lives (x) and (y) survive after t years.

The joint life status: Independence

Alternative expression for the distribution function:

$$F_{T_{xy}}(t) = F_x(t) + F_y(t) - F_{T_xT_y}(s, t)$$

In the case where T_x and T_y are independent:

$$\begin{aligned} {}_t p_{xy} &= P\left[T_x > t, T_y > t\right] \\ &= P\left(T_x > t\right) \times P\left(T_y > t\right) \\ &= {}_t p_x \times {}_t p_y \end{aligned}$$

$${}_t q_{xy} = {}_t q_x + {}_t q_y - {}_t q_x \; {}_t q_y$$

Remember !!! (**even in the case of independence**):
${}_t q_{xy} \neq {}_t q_x + {}_t q_y$

The last survivor status

- Status that survives so long as there is at least one member alive, and therefore fails upon the last death.
- Notation: $(\overline{xy})$
- For two lives: $T_{\overline{xy}} = \max(T_x, T_y)$

The last survivor status

General relationship among T_{xy}, $T_{\overline{xy}}$, T_x, and T_y:

$$T_{xy} + T_{\overline{xy}} = T_x + T_y$$

$$T_{xy} \times T_{\overline{xy}} = T_x \times T_y$$

$$a^{T_{xy}} + a^{T_{\overline{xy}}} = a^{T_x} + a^{T_y} \text{ for any } a > 0$$

For each outcome, note that T_{xy} is equal either T_x or T_y, and therefore, $T_{\overline{xy}}$ equals the other.

The last survivor status distribution

Distribution of $T_{\overline{xy}}$
Recall:

$$Pr[A \cup B] + Pr[A \cap B] = Pr[A] + Pr[B]$$

Choose events $A = \{T_x \leq t\}$ and $B = \{T_y \leq t\}$ so that

$$\begin{aligned} A \cup B &= T_{xy} \\ A \cap B &= T_{\overline{xy}} \end{aligned}$$

This leads to:

$$F_{T_{xy}}(t) + F_{T_{\overline{xy}}}(t) = F_x(t) + F_y(t)$$

The last survivor status distribution

$$S_{T_{xy}}(t) + S_{T_{\overline{xy}}}(t) = S_x(t) + S_y(t)$$

$$\begin{aligned} {}_tp_{xy} + {}_tp_{\overline{xy}} &= {}_tp_x + {}_tp_y \Leftrightarrow \\ {}_tp_{\overline{xy}} &= {}_tp_x + {}_tp_y - {}_tp_{xy} \end{aligned}$$

$$f_{T_{xy}}(t) + f_{T_{\overline{xy}}}(t) = f_x(t) + f_y(t)$$

These relationships lead to finding the distribution of $T_{\overline{xy}}$.

$$F_{T_{\overline{xy}}}(t) = F_x(t) + F_y(t) - F_{T_{xy}}(t) = F_{T_x T_y}(t)$$

which is obvious from $F_{T_{\overline{xy}}}(t) = P(T_x \leq t \cap T_y \leq t])$.

The last survivor status distribution

Interpretation of Survival function

$$S_{T_{\overline{xy}}}(t) = {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}$$

$$\Leftrightarrow$$

$$S_{T_{\overline{xy}}}(t) = {}_tp_x\, {}_tp_y + {}_tp_x\, (1 - {}_tp_y) + {}_tp_y\, (1 - {}_tp_x)$$

- ${}_tp_x\, {}_tp_y$ means that both x and y alive after t years
- ${}_tp_x\, (1 - {}_tp_y)$ means that x is alive and y is dead after t years
- ${}_tp_y\, (1 - {}_tp_x)$ means that y is alive and x is dead after t years

Interpretations of probabilities

${}_tp_{xy}$ is the probability that both lives (x) and (y) will be alive after t years.

${}_tp_{\overline{xy}}$ is the probability that at least one of lives (x) and (y) will be alive after t years.

In contrast:

${}_tq_{xy}$ is the probability that at least one of lives (x) and (y) will be dead within t years.

${}_tq_{\overline{xy}}$ is the probability that both lives (x) and (y) will be dead within t years.

Force of mortality for joint life status

Define the force of mortality (similar manner to any random variable):

$$\mu_{x+t:y+t} = \frac{f_{T_{xy}}(t)}{1 - F_{T_{xy}}(t)}$$

$$= \frac{f_{T_{xy}}(t)}{S_{T_{xy}}(t)} = \frac{f_{T_{xy}}(t)}{{}_t p_{xy}}$$

Then the pdf of T_{xy} :

$$f_{T_{xy}}(t) = {}_t p_{xy} \times \mu_{x+t:y+t}$$

Force of mortality for joint life status

In the case of independence

$$f_{T_{xy}}(t) = \ {}_tp_x \ {}_tp_y \left(\mu_{x+t} + \mu_{y+t}\right)$$

to be shown in details in MTH6157
Hence,

$$\mu_{x+t:y+t} = \mu_{x+t} + \mu_{y+t}$$

The force of mortality of the joint life status is the sum of the individuals' force of mortality, when lives are independent.
Note that the force of mortality of the joint life status is also called **the "force of failure"** of the joint life status

Force of mortality for life survivor status

Similarly:

$$\mu_{\overline{x+t:y+t}} = \frac{f_{T_{\overline{xy}}}(t)}{1 - F_{T_{\overline{xy}}}(t)} = \frac{f_{T_{\overline{xy}}}(t)}{S_{T_{\overline{xy}}}(t)} = \frac{f_{T_{\overline{xy}}}(t)}{{}_tp_{\overline{xy}}}$$

Then the pdf of T_{xy} :

$$f_{T_{\overline{xy}}}(t) = {}_tp_{\overline{xy}} \times \mu_{\overline{x+t:y+t}}$$

Just as we used l_x d_x q_x from life tables for calculations involving a single life, so we also have l_{xy} d_{xy} q_{xy} which can also be written $l_{x:y}$ $d_{x:y}$ $q_{x:y}$ for extra clarity
As,

$$_tp_{xy} = \ _tp_x \ _tp_y$$

we know that

$$l_{xy} = l_x l_y$$

and

$$_tp_{xy} = \frac{l_{x+t:y+t}}{l_{x:y}}$$

and

$$d_{xy} = l_{xy} - l_{x+1:y+1}$$

so

$$q_{xy} = \frac{d_{xy}}{l_{xy}}$$

Example

The table below shows extracts from two lifes tables appropriate for a husband and wife who are assumed to be independent with respect to mortality:

Male		Female	
x	l_x	y	l_y
65	43302	60	47260
66	42854	61	47040
67	42081	62	46755
68	41351	63	46500
69	40050	64	46227

Example

a) Calculate ${}_3p_{xy}$ for a husband aged $x = 66$ and a wife aged $y = 60$.

$$
\begin{aligned}
{}_3p_{xy} &= {}_3p_x \times {}_3p_y \\
&= \frac{40050}{42854} \times \frac{46500}{47260} \\
&= 0.9195
\end{aligned}
$$

Example

b) Calculate ${}_2p_{\overline{xy}}$ for a husband aged $x = 65$ and a wife aged $y = 62$.

$$
\begin{aligned}
{}_2p_{\overline{xy}} &= {}_2p_x + {}_2p_y - {}_2p_x\, {}_2p_y \\
&= \frac{42081}{43302} + \frac{46227}{46755} - \frac{42081}{43302} \times \frac{46227}{46755} \\
&= 0.9997
\end{aligned}
$$

Example

c) Calculate the probability the probability that a husband, currently aged 65, dies within two years and that his wife, currently aged 61, survives at least two years.
Since the two lives are indpendent with respect to mortality, the required probability is:

$$(1 - {}_2p_x)\ {}_3p_y = 0.0279$$

Curtate Joint Life

K_{xy} is integer part of T_{xy} - the discrete random variable which measures the curtate joint future lifetime of x and y

The probability function of K_{xy} is given by

$$
\begin{aligned}
P[K_{xy} &= k] = P[k \leq T_{xy} \leq k+1] \\
&= F_{T_{xy}}(k+1) - F_{T_{xy}}(k) \\
&= (1 - {}_{k+1}p_{xy}) - (1 - {}_{k}p_{xy}) \\
&= {}_{k}p_{xy} - {}_{k+1}p_{xy} \\
&= {}_{k}p_{xy} - {}_{k}p_{xy}\ p_{x+k:y+k} \\
&= {}_{k}p_{xy}\,(1 - p_{x+k:y+k}) \\
&= {}_{k}p_{xy}\ q_{x+k:y+k} \\
&= {}_{k|}q_{xy}
\end{aligned}
$$

Curtate Last Survivor lifetime

The curtate last survivor lifetime of (x) and y is $K_{\overline{xy}}$ - the integer part of $T_{\overline{xy}}$
$K_{\overline{xy}}$ cumulative distribution function is:

$$\begin{aligned}
P[K_{\overline{xy}} &= k] = P[k \leq T_{\overline{xy}} \leq k+1] \\
&= F_{T_{\overline{xy}}}(k+1) - F_{T_{\overline{xy}}}(k) \\
&= F_{T_x}(k+1) + F_{T_y}(k+1) - F_{T_{xy}}(k+1) \\
&\quad - \left(F_{T_x}(k) + F_{T_y}(k) - F_{T_{xy}}(k)\right) \\
&= P[K_x = k] + P[K_y = k] - P[K_{xy} = k] \\
&= {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}
\end{aligned}$$

Whole Life Insurance discrete case

Consider an insurance under which the benefit of \$1 is paid at the end of the year of failure of status u.
Status u could be any **joint life** or **last survivor** status e.g. xy, $\overline{xy}$.

- the time at which the benefit is paid: $K_u + 1$
- the present value (at issue) of the benefit is: $Z = v^{K_u+1}$

Expected present value of benefits

$$A_u = E\left(v^{K_u+1}\right) = \sum_{k=0}^{\infty} v^{k+1} P\left[K_u = k\right] = \sum_{k=0}^{\infty} v^{k+1} \; {}_{k|}q_u$$

Variance of benefits: $Var\left(v^{K_u+1}\right) = {}^2A_u - (A_u)^2$

Whole Life Insurance continous case

Consider an insurance under which the benefit of \$1 is paid immediatly at the end (failure) of status u.
Status u could be any **joint life** or **last survivor** status e.g. xy, $\overline{xy}$.

- the time at which the benefit is paid: T_u
- the present value (at issue) of the benefit is: $Z = v^{T_u}$

Expected present value of benefits

$$\bar{A}_u = E\left(v^{T_u}\right) = \int_0^\infty v^t \ {}_tp_u \ \mu_{u+t} \ dt$$

Variance of benefits: $Var\left(v^{T_u}\right) = {}^2\bar{A}_u - \left(\bar{A}_u\right)^2$

Whole Life Insurance illustrations

$$A_{xy} = \sum_{k=0}^{\infty} v^{k+1} \ {}_{k|}q_{xy} = \sum_{k=0}^{\infty} v^{k+1} \ {}_{k}p_{xy} \ q_{x+k:y+k}$$

$$A_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \ {}_{k|}q_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \ \left({}_{k|}q_{x} + \ {}_{k|}q_{y} - \ {}_{k|}q_{xy} \right)$$

$$\bar{A}_{xy} = \int_0^{\infty} v^{t} \ {}_{t}p_{xy} \ \mu_{x+t:y+t} \ dt$$

$$\begin{aligned} \bar{A}_{\overline{xy}} &= \int_0^{\infty} v^{t} \ {}_{t}p_{\overline{xy}} \ \mu_{\overline{x+t:y+t}} \ dt \\ &= \int_0^{\infty} v^{t} \ \left(\ {}_{t}p_{x} \ \mu_{x+t} + \ {}_{t}p_{y} \ \mu_{y+t} - \ {}_{t}p_{xy} \ \mu_{x+t:y+t} \ \right) dt \end{aligned}$$

Whole Life Insurance illustrations

Useful relations:

$$A_{xy} + A_{\overline{xy}} = A_x + A_y$$

$$\bar{A}_{xy} + \bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y$$

Note in the discrete case:

$${}_{k|}q_{xy} = {}_{k}p_{xy}\left(1 - p_{x+k:y+k}\right) = {}_{k}p_{xy} - {}_{k+1}p_{xy}$$

Annuities benefits - discrete

Consider an n-year temporary (term) life annuity-due on status u.
In single life setting: a temporary life annuity makes regular payments to the annuitant until death if death is before n or until the set expiration date if the annuitant is alive.
Moving to joint life:
The present value at the issue of the benefit:

$$Y = \begin{cases} \ddot{a}_{\overline{K_u+1}|} \text{ if } K_u < n \\ \ddot{a}_{\overline{n}|} \text{ if } K_u \geq n \end{cases}$$

Annuities benefits - discrete

Expected present value of benefits for temporary annuity:

$$\ddot{a}_{u:\overline{n}|} = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \ {}_{k|}q_u + \ddot{a}_{\overline{n}|} \ {}_{n}p_u$$

Remember in the case of single life:

$$\ddot{a}_x = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \ {}_{k|}q_u \equiv \sum_{k=0}^{n-1} v^{k+1} \ {}_{k}p_x$$

Annuities benefits - discrete

$$\ddot{a}_{u:\overline{n}|} = \sum_{k=0}^{n-1} v^k \ {}_k p_u$$

or:

$$\ddot{a}_{u:\overline{n}|} = \frac{1}{d}\left(1 - A_{u:\overline{n}|}\right)$$

Variance of benefits

$$Var\left(Y\right) = \frac{1}{d^2}\left({}^2A_{u:\overline{n}|} - \left(A_{u:\overline{n}|}\right)^2\right)$$

Annuities benefits - discrete

- $\ddot{a}_{xy:\overline{n}|}$ - each payment made only if the lives (x) and (y) are alive at the time the payment is due.
- $\ddot{a}_{\overline{xy}:\overline{n}|}$ - each payment made only if at least one of the lives (x) and (y) are alive at the time the payment is due.

Annuities benefits - discrete

Using the earlier example (page 23) find $\ddot{a}_{xy:\overline{5}|}$ and $\ddot{a}_{\overline{xy}:\overline{5}|}$

$$\ddot{a}_{xy:\overline{5}|} = \sum_{t=0}^{4} v^t \; {}_tp_{xy}$$

with $v = \frac{1}{1.05}, x = 65, y = 60.$

$$\begin{aligned} \ddot{a}_{xy:\overline{5}|} &= 4.3661 \\ \ddot{a}_{\overline{xy}:\overline{5}|} &= 4.5437 \end{aligned}$$

Note that $\ddot{a}_{xy:\overline{5}|} \leq \ddot{a}_{\overline{xy}:\overline{5}|}$ as $T_{xy} \leq T_{\overline{xy}}$

Annuities benefits - continous

Consider an annuity for which the benefit of $1 is paid each year continuously as long as a status u continues.
The present value (at issue) of the benefit: $Y = \bar{a}_{\overline{T_u}|}$
Expected present value of benefits: $\bar{a}_u = \int_0^\infty \bar{a}_{\overline{t}|} \; {}_tp_u \; \mu_{u+t} \; dt = \int_0^\infty v^t \; {}_tp_u \; dt$
Variance of benefits: $\frac{1}{d^2}\left({}^2\bar{A}_u - (\bar{A}_u)^2\right)$

Annuities

Useful relations

$$\ddot{a}_{xy} + \ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y$$
$$\bar{a}_{xy} + \bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y$$

Different types of annuities

Type of life annuity	Single life x	Joint life status xy	Last survivor status $\overline{xy}$
Whole life a-due	$\ddot{a}_x$	$\ddot{a}_{xy}$	$\ddot{a}_{\overline{xy}}$
Whole life a-immediate	a_x	a_{xy}	$a_{\overline{xy}}$
Temporary life a-due	$\ddot{a}_{x:\overline{n}\rceil}$	$\ddot{a}_{xy:\overline{n}\rceil}$	$\ddot{a}_{\overline{xy}:\overline{n}\rceil}$
Temporary life a-immediate	$a_{x:\overline{n}\rceil}$	$a_{xy:\overline{n}\rceil}$	$a_{\overline{xy}:\overline{n}\rceil}$
Whole life a-continuous	$\bar{a}_x$	$\bar{a}_{xy}$	$\bar{a}_{\overline{xy}}$
Temporary life a-continuous	$\bar{a}_{x:\overline{n}\rceil}$	$\bar{a}_{xy:\overline{n}\rceil}$	$\bar{a}_{\overline{xy}:\overline{n}\rceil}$

Type of life insurance	Single life x	Joint life status xy	Last survivor status $\overline{xy}$
Whole life - discrete	A_x	A_{xy}	$A_{\overline{xy}}$
Whole life - continuous	$\bar{A}_x$	$\bar{A}_{xy}$	$\bar{A}_{\overline{xy}}$
Term - discrete	$A^{\,1}_{x:\overline{n}\vert}$	$A^{\;1}_{\overline{xy}:\overline{n}\vert}$	$A^{\;1}_{\overline{xy}:\overline{n}\vert}$
Term - continuous	$\bar{A}^{\,1}_{x:\overline{n}\vert}$	$\bar{A}^{\;1}_{\overline{xy}:\overline{n}\vert}$	$\bar{A}^{\;1}_{\overline{xy}:\overline{n}\vert}$
Endowment - discrete	$A_{x:\overline{n}\vert}$	$A_{xy:\overline{n}\vert}$	$A_{\overline{xy}:\overline{n}\vert}$
Endowment - continuous	$\bar{A}_{x:\overline{n}\vert}$	$\bar{A}_{xy:\overline{n}\vert}$	$\bar{A}_{\overline{xy}:\overline{n}\vert}$
Pure endowment	$A_{x:\overset{1}{\overline{n}\vert}}$ or ${}_nE_x$	$A_{xy:\overset{1}{\overline{n}\vert}}$ or ${}_nE_{xy}$	$A_{\overline{xy}:\overset{1}{\overline{n}\vert}}$ or ${}_nE_{\overline{xy}}$

Premiums and Reserves

Similar to single life contracts
Example:
An insurer sells an annuity policy to Shannon and Riley who are both aged 60, with the following premium and benefits:

- Level annual premiums are payable for at most 10 years, while both Shannon and Riley are alive.
- There are no annuity payments during the first 10 years
- After 10 years, at the start of each year the annuity pays:
  - $\$120,000$ if both Shannon and Riley are alive at the payment date, and
  - $\$70,000$ if only one of them is alive at the payment date

Assume independent future lifetimes and the mortality follows the Standard ultimate life table with $i = 5\%$.
Calculate the net premium

Premiums and Reserves

EPV of premiums(income)=EPV of annuity payment (outgoings)

Let P be the annual premium. The EPV of premiums is:

$$P\ddot{a}_{60:60:\overline{10}|} = 7.8080P$$

The EPV of the annuity payment:

$$
\begin{aligned}
& 120,000 \sum_{t=10}^{\infty} v^t \;{}_tp_{60:60} + 2 \times 70,000 \sum_{t=10}^{\infty} v^t \;{}_tp_{60} \left(1 - \;{}_tp_{60}\right) \\
= \;& 120,000 \;{}_{10|}\ddot{a}_{60:60} + 140,000\left({}_{10|}\ddot{a}_{60} - {}_{10|}\ddot{a}_{60:60}\right) \\
= \;& 140,000 \;{}_{10|}\ddot{a}_{60} - 20,000\,{}_{10|}\ddot{a}_{60:60}
\end{aligned}
$$

${}_{10|}\ddot{a}_{60} = {}_{10}E_{60}\ddot{a}_{70} = 6.9485$

${}_{10|}\ddot{a}_{60:60} = \ddot{a}_{60:60} - \ddot{a}_{60:60:\overline{10|}} = 5.4417$

Hence $P = \$110,650$

Premiums and Reserves

However the situation for last survivor assurances is more complex and particular care is needed

- reserve will be different depending on whether both lives remain alive or one has already died.
- in both these cases, the last survivor contract is still in force and the premium payable remains the same
- but the reserve jumps on the first death

Last Survivor Reserves

For simplicity let us consider the prospective net premium reserve for a whole life, last survivor assurance:
Whilst both x and y are alive, the reserve is:

$$ {}_tV_{\overline{x:y}} = A_{\overline{x+t:y+t}} - P_{\overline{x:y}}\ddot{a}_{\overline{x+t:y+t}} $$

Once say y has died, the reserve is

$$ {}_tV_{\overline{x:y}} = A_{x+t} - P_{\overline{x:y}}\ddot{a}_{x+t} $$

which is significantly larger

Contingent Functions

It is possible to compute probabilities, insurances and annuities based on the failure of the status that is contingent on the order of the deaths of the members in the group, e.g. (x) dies before (y). These are called contingent functions.
Consider the probability that (x) dies before (y) - assuming independence:

$$
\begin{aligned}
P\left[T_x < T_y\right] &= \int_0^\infty f_{T_x}(t)\, S_{T_y}(t)\, dt \\
&= \int_0^\infty {}_t p_x \; \mu_{x+t} \; {}_t p_y \, dt \\
&= \int_0^\infty {}_t p_{xy} \; \mu_{x+t} dt
\end{aligned}
$$

Contingent Functions

- ${}_nq_{\overset{1}{x}y}$ is the probability that (x) dies before (y) and within n years

$$ {}_nq_{\overset{1}{x}y} = \int_0^n {}_tp_{xy}\ \mu_{x+t} dt $$

- ${}_nq_{x\overset{1}{y}}$ is the probability that (y) dies before (x) and within n years

$$ {}_nq_{x\overset{1}{y}} = \int_0^n {}_tp_{xy}\ \mu_{y+t} dt $$

Note

$$ {}_nq_{\overset{1}{x}y} + {}_nq_{x\overset{1}{y}} = 1 $$

Similarlly ${}_nq_{\overset{2}{x}y}$ is the probability that (x) dies after (y) and within n years and ${}_nq_{x\overset{2}{y}}$ is the probability that (y) dies after (x) and within n years

Contingent Functions

An insurance of \$1 is payable immediately on the death of (x) provided that (y) is still alive. The present value is: 0 if $T_x > T_y$ and v^{T_x} if $T_x \leq T_y$ The expected present value of this insurance is denoted by $\bar{A}^1_{xy}$.

$$\bar{A}^1_{xy} = \int_0^{\infty} v^t \ {}_t p_{xy} \ \mu_{x+t} dt$$

If the benefit is payable at the end of the year of death rather than immediately, the corresponding expected present value is found by summing rather than integrating:

$$\bar{A}^1_{xy} = \sum_0^{\infty} v^{t+1} \ {}_t p_{xy} \ q^1_{x+t:y+t}$$

Contingent Functions

An insurance of $1 is payable at the moment of death of (y) if predeceased by (x), i.e. if (y) dies after (x). The expected present value of this insurance is denoted by $\bar{A}_{xy}{}^{2}$. Assume (x) and (y) are independent.

$$\bar{A}_{xy}{}^{2} = \bar{A}_{y} - \bar{A}^{1}_{xy}$$

$$\bar{A}_{xy}{}^{2} = \int_{0}^{\infty} v^{t} \bar{A}_{y+t} \; {}_{t}p_{xy} \; \mu_{x+t} dt$$

Example: Premiums

Tom and John are both aged 75 with independent future lifetime. They purchase an insurance policy which pays $\$100,000$ immediately on the death of John provided he dies after Tom. Premiums are payable continuously at a rate P per year while both lives are alive. You are given that $\bar{A}_{75} = 0.46570$ and $\bar{A}_{75:75} = 0.57481$ and $i = 6\%$. Calculate P.

Example: Premiums

As P is the total premium each year, the EPV of premiums is

$$P\,\bar{a}_{75:75} = P\left(\frac{1-\bar{A}_{75:75}}{\delta_{6\%}}\right) = 7.2970\,P.$$

Example: Premiums

The EPV of the death benefit is

$$100\,0000\left(\bar{A}_{75} - \bar{A}^{\;1}_{75:75}\right)$$

since the benefit is payable on John's death, but not if he dies first. As

$$\bar{A}^{\;1}_{75:75} + \bar{A}_{75:\overset{1}{75}} = \bar{A}_{75:75}\ ,$$

the EPV of the death benefit is

$$100\,0000\left(0.46570 - \tfrac{1}{2}\,(0.57481)\right) = 17\,829.50,$$

and so $P = \$2\,443.39$.

Reversionary annuity

The simplest form is: an annuity that begins on the death of (x) if (y) is then alive, and continues during the remaining lifetime of (y)

- (x) is the 'counter life' or 'failing life'
- (y) is the annuitant

Notation:

- $\bar{a}_{x|y}$ - if payable continuously immediately on the death of (x), or
- $a_{x|y}$ - if payable annually in arrears from the end of the year of death of (x)

Reversionary annuity

an annuity of \$1 per year payable continuously to a life now aged y, commencing at the moment of death of (x) - briefy annuity to (y) after (x).

Simple way of calculating reversionary annuities:

$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$

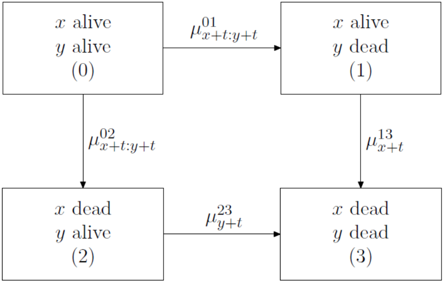
$a_{x|y} = a_y - a_{xy} = \frac{A_{xy} - A_y}{d}$

Example - for next class

A simple pension arrangement offers a pension of $\$9,000$ per annum payable monthly in arrears from age 65 plus a widow or widower's pension of 50% of the original pension amount again payable monthly in arrears. This pension is to be funded by level annual-in-advance contributions up to age 65. Robbie is currently age 55 and his wife Sara is exactly five years younger. If the present value of Robbie's current pension savings is $\$23,000$ how much should he now contribute per annum into this pension arrangement assuming 4% per annum interest and mortality using AM92 Ultimate and $\ddot{a}_{65:60} = 8.235$?

A different setting: multiple state

We need to evaluate probabilities of survival/death for our two lives.



- $\mu^{01}_{x+t:y+t}$ is the intensity of mortality for (y) she is aged $y+t$ given that (x) is still alive and aged $x+t$
- if (x), has died, the intensity of mortality for (y) depends on her current age, and the fact that (x) has died, but not on how long (x) has been dead: μ^{23}_{y+t}

Translating the probabilities/forces earlier defined, the following should now be straightforward to verify:

$$
\begin{aligned}
{}_t p_{xy} &= {}_t p_{xy}^{00} \\
{}_t q_{xy} &= {}_t p_{xy}^{01} + {}_t p_{xy}^{02} + {}_t p_{xy}^{03} \\
{}_t p_{\overline{xy}} &= {}_t p_{xy}^{00} + {}_t p_{xy}^{01} + {}_t p_{xy}^{02} \\
{}_t q_{\overline{xy}} &= {}_t p_{xy}^{03}
\end{aligned}
$$

$$
{}_t q_{xy}^{1} = \int_0^t {}_s p_{xy}^{00} \mu_{x+s:y+s}^{02} ds
$$

$$
{}_t q_{xy}^{2} = \int_0^t {}_s p_{xy}^{01} \mu_{x+s}^{03} ds
$$

Multiple state framework

In terms of the annuity functions, the following should also be straightforward to verify:

- $\bar{a}_{xy} = \bar{a}^{00}_{xy} = \int_0^\infty e^{-\delta t} \, {}_t p^{00}_{xy} dt$
- $\bar{a}_{\overline{xy}} = \bar{a}^{00}_{xy} + \bar{a}^{01}_{xy} + \bar{a}^{02}_{xy} = \int_0^\infty e^{-\delta t} \left({}_t p^{00}_{xy} + {}_t p^{01}_{xy} + {}_t p^{02}_{xy} \right) dt$
- $\bar{a}_{x|y} = \bar{a}^{02}_{xy} = \int_0^\infty e^{-\delta t} \, {}_t p^{02}_{xy} dt$

Multiple state framework

In terms of insurance functions, the following should also be straightforward to verify:

- $\bar{A}_{xy} = \displaystyle\int_0^\infty \mathrm{e}^{-\delta t}\ {}_tp_{xy}^{00}\left(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02}\right)dt$
- $\bar{A}_{\overline{xy}} = \displaystyle\int_0^\infty \mathrm{e}^{-\delta t}\left({}_tp_{xy}^{01}\ \mu_{x+t}^{13} + {}_tp_{xy}^{02}\ \mu_{y+t}^{23}\right)dt$
- $\bar{A}_{xy}^{1} = \displaystyle\int_0^\infty \mathrm{e}^{-\delta t}\ {}_tp_{xy}^{00}\ \mu_{x+t:y+t}^{02}dt$
- $\bar{A}_{xy}^{2} = \displaystyle\int_0^\infty \mathrm{e}^{-\delta t}\ {}_tp_{xy}^{01}\ \mu_{x+t}^{13}dt$