Actuarial Mathematics II
MTH5125

Reserves

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Plan

- Based on Chapter 7, DHW
- Why hold reserves?
- Future loss random variable (i.e. net future loss) at time t
- Reserves/policy values determination (prospective method)
- Recursive formula for policy values (discrete and continuous)
- Retrospective policy values

Reserves (Policy Values)

- Reserves: Money set aside to be able to cover insurer's future financial obligations as promised through the insurance contract.
- Reserves are also called provisions (mostly in Europe) or policy values
- An actuary working in a Life Office is responsible for preparing an Actuarial Opinion and Memorandum: that the company's assets are sufficient to back reserves.

Why hold reserves?

- The expected cost of paying the benefits generally increases over the contract term; but the periodic premiums used to fund these benefits are level.
- The portion of the premiums not required to pay expected cost in the early years are therefore set aside (or provisioned) to fund the expected shortfall in the later years of the contract.
- Reserves also help reduce cost of benefits as they also earn interest while being set aside.
- Reserves are usually held on a per-contract basis, it is the responsibility of the actuary to ensure that in the aggregate, the company's assets are enough to back these reserves.

Future loss random variable (future random loss)

- When we discussed the setting of premium, we made use of future loss random variables.
- In that context, we only considered future loss random variables at issue, i.e., future random loss at time 0.
- We're again to make use of future loss random variables in order to study policy values (or reserves).

Future loss random variable (future random loss)

The insurer's **future loss at time** t: PV_t of future benefits outgo (claims) $-$ PV_t of future premiums
Thus, the future loss is only concerned with events happening **after time** t — benefits outgo and premiums occurring prior to time t do not affect this calculation.
Net/Gross Random Future Loss at Time t

$$L_t^n = PV_t\,(\text{benefits}) - PV_t\,(\text{net premiums})$$

$$L_t^g = PV_t\,(\text{benefits}) + PV_t\,(\text{expenses}) - PV_t\,(\text{gross premiums})$$

The notation $PV_t(X)$ denotes the present value at time t of X.

Future loss random variable - Example 1

Consider a 40-year old that has purchased a whole life insurance policy with \$100,000 payable at the end of the year of death. Premiums are payable at the beginning of the year. Using the Standard Ultimate Mortality Model with $i = 5\%$ gives $A_{40} = 0.12106$ and $A_{50} = 0.18931$.
First find the annual premium.

$$\begin{aligned} L_0^n &= 100,000 v^{K_{40}+1} - P\ddot{a}_{\overline{K_{40}+1}|} \\ E\left(L_0^n\right) &= 100,000 A_{40} - P\ddot{a}_{\overline{40}|} = 0 \end{aligned}$$

$$\begin{aligned} P &= 100,000 \frac{A_{40}}{\ddot{a}_{40}} = 100,000 \frac{0.05}{1.05} \frac{A_{40}}{1 - A_{40}} \\ &= 655.8766 \end{aligned}$$

We used: $A_x = 1 - d\ddot{a}_x$
What is the net random loss at $t = 10$?

Future loss random variable - Example 1

What is the net random loss at $t = 10$?

$$L^n_{10} = 100,000v^{K_{50}+1} - P\ddot{a}_{\overline{K_{50}+1}|}$$

$$\begin{aligned} E\left(L^n_{10}\right) &= 100,000A_{50} - 655.8766\ddot{a}_{\overline{50}|} \\ &= 100,000A_{50} - 655.8766\frac{1 - A_{50}}{0.05/1.05} \\ &= 7,765.03 \end{aligned}$$

Unlike at issue, this expected future loss is not zero; the future premiums are not expected to be sufficient to cover future benefits.

The insurer would need to have, on average, \$7,765.03 on hand in addition to future premiums in order to cover the future benefits

Future loss random variable - Example 2

Let's look at a 20-year term insurance issued to (50). The sum insurred is \$500, 000, level annual premiums are payable troughout the term. Basis is Standard Ultimate Mortality Model with $i = 5\%$. Equation of value (equivalence principle):

$$500,000A^{1}_{50:\overline{20|}} - P\ddot{a}_{50:\overline{20|}} = 0$$

Reminder:

$$\begin{aligned} A^{1}_{50:\overline{20|}} &= A_{50} - v^{20}\ {}_{20}p_{50}A_{70} \\ &= A_{50} - \ {}_{20}E_{50}A_{70} \\ &= 0.18931 - 0.34824 \times 0.42818 \\ &= 0.0402005968 \end{aligned}$$

$\ddot{a}_{50:\overline{20|}} = \ddot{a}_{50} - {}_{20}E_{50}\ddot{a}_{70} = 17.0245 - 0.34824 \times 12.0083 =$ 0.1284275

$$P = \frac{500,000A^{1}_{50:\overline{20|}}}{\ddot{a}} = 1,565.11$$

Future loss random variable - Example 2

If we were to plot EPV at the start of each year of premiums minus claims in that year:

$$P - 500,000vq_{50+t} \text{ where } t = 0, 1, .., 19$$

For example for $t = 0$ the above EPV is $1565.11 - 492.04 = 1073.07$ and so on.

Future loss random variable - Example 2

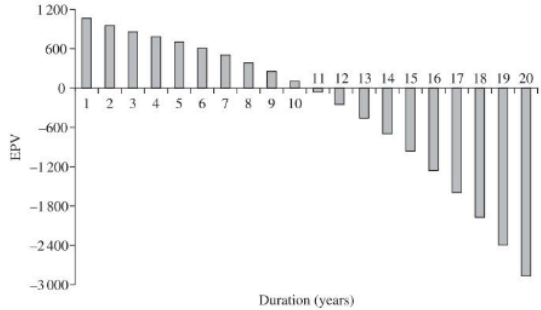

Figure 7.2 EPV at the start of each year of premiums minus claims in that year for a 20-year term insurance, sum insured $500 000, issued to (50).

We see a positive surplus in early years which can be used to build up the insurer's assets.

These assets will be needed in the later years when the premium is not sufficient to pay for the expected benefits.

Prospective Policy Values

The amount needed to cover the shortfall between future benefits and future premiums ($\$7,765$ in the Example 1) is called the **policy value at time** t and is denoted generically by ${}_tV$.

The process of calculating policy values is known as valuation. This is also called reserving, and policy values are known as reserves.

The general "prospective" formula for a policy value is

$$ {}_tV = EPV_t\left(\text{future benefits}\right) - EPV_t\left(\text{future premiums}\right) = E\left(L_t\right) $$

Policy Values Basis

- The assumptions (such as mortality, expenses, interest, etc.) used in a policy value calculation form **the policy value basis**.
- The assumptions used to originally calculate the premiums for the policy form **the premium basis**.
- In practice, they tend to differ.

The policy value basis used will typically depend on the purpose of the valuation.

Some common reasons for valuations:

- Internal management information
- Regulatory requirements
- Shareholder reporting
- Reporting required for taxation purposes

Policy Values

The **gross premium policy value at time** t is the expected value (at time t) of the gross future loss random variable.

- The premiums used in the calculation are the **actual premiums** for the policy.

The **net premium policy value at time** t is the expected value (at time t) of the net future loss random variable.

- The premiums used in the calculation are the **net premiums calculated by the Equivalence Principle**, applied at the age of policy issue, calculated on the policy value basis.
- **No expenses are taken into account.**

Policy Values - Example 3

Consider a whole life policy for a life aged 50 with $\$100,000$ death benefit payable at the end of the year of death. Gross premiums of $\$1,370$ are paid annually in advance.

1. Calculate the gross premium policy value 5 years after the inception of the contract, assuming the policy is still in force using the following basis:

- Standard Select Survival Model $i = 5\%$(Table D, this is a 2-year select model). Expenses are 12.5% of **each premium**

2. Calculate the net premium policy value at time 5, using the same basis as above, but $i = 4\%$. You are given $A_{[50]} = 0.25570$ and $A_{55} = 0.30560$

Policy Values - Example 3

We assume that the life is select at age 50, when the policy is purchased. At duration 5 the life is 55 and no longer select as this is **a 2 year select model.** Note that a premium due at age 55 is regarded as a future premium in the calculation of policy values. The gross future random loss at $t = 5$ is:

$$L_5^g = 100,000v^{K_{55}+1} - 0.875 \times 1,370\ddot{a}_{\overline{K_{55}+1}|}$$

$$\begin{aligned} E\left(L_5^g\right) &= {}_5V^g = 100,000A_{55} - 0.875 \times 1,370\ddot{a}_{\overline{55}|} \\ &= 100,000 \times 0.23524 - 0.875 \times 1,370 \times 16.0599 \\ &= 4272.195 \end{aligned}$$

Policy Values - Example 3

For the net premium policy values we recalculate the net premium for the contract on a **different policy value basis**.
Equivalent principle:

$$E\left(Sv^{K_{[50]}+1}\right) = P\left(\ddot{a}_{\overline{K_{[50]}+1|}}\right)$$

$$SA_{[50]} = P\ddot{a}_{[50]}$$

$$\ddot{a}_{[50]} = \frac{1 - A_{[50]}}{d} = \frac{1 - 0.25570}{0.04/1.04} = 19.3518$$

$$P = 100,000\frac{A_{[50]}}{\ddot{a}_{[50]}} = 1321.32$$

Policy Values - Example 3

$$\ddot{a}_{55} = \frac{1 - A_{55}}{d} = \frac{1 - 0.30560}{0.04/1.04} = 18.0544$$

$$_5V^n = E\left(L_5^n\right) = 100,000A_{55} - 1321.32\ddot{a}_{55} = 6,704.29$$

In this example the net premium calculations ignores expenses, but uses a lower interest rate and a lower premium than the gross premium policy value, both of which provides a margin allowing implicitly for expenses.

The policy under the net premium calculation indicates that the insurer needs to hold more reserves for the policy than the reserves under the gross premium calculations.

Recursive formulae for policy values

We can develop some useful recursive formulas for policy values, that is, formulae relating a policy value at time t to the policy value for the same policy value at time $t+1$.
We can thus calculate policy values without having to start from scratch every time.

1. Write out the policy value at time t in terms of the EPV for all future policy cash flows.
2. Split the future cash flow values into those occurring in the first year and those occurring in later years.
3. Regroup the EPV for later year cash flows in terms of a policy value at time $t+1$.

Recursive formulae for policy values

We use the previous example to calculate ${}_6V^g$

$$
\begin{aligned}
{}_5V^g &= 100,000A_{55} - 0.875 \times 1,370\ddot{a}_{\overline{55}|} \\
&= \underbrace{100,000A_{55} + 0.125 \times 1,370\ddot{a}_{\overline{55}|}}_{\text{Benefits outgo +expenses}} - \underbrace{1,370\ddot{a}_{\overline{55}|}}_{\text{Premiums (income)}}
\end{aligned}
$$

Recursive formulae for policy values

Looking forward at $t = 6$

$$\left({}_5V^g + 1370 - 0.125 \times 1,370\right)(1+i) = \underbrace{100,000q_{55}}_{\text{benefit paid if dead}} + \underbrace{p_{55}\ {}_6V^g}_{\text{reserve if alive}}$$

We need one additional piece of information, namely $q_{55} = \frac{l_{55}-l_{56}}{l_{55}} \frac{97846.20-97651.21}{97846.20} = 0.00199.$

$$\begin{aligned} {}_6V^g &= \frac{\left({}_5V^g + 1370 - 0.125 \times 1,370\right)(1+i) - 100,000q_{55}}{1-q_{55}} \\ {}_6V^g &= 8,035.88 \end{aligned}$$

Recursive formulae for policy values

Using the following notations:
P_t : premium payable at time t
e_t : expenses payable at time t
S_{t+1} : death benefit payable at time $t+1$ if the insured dies during the year
E_{t+1} : termination related expenses at time $t+1$
i_t : annual effective interest rate in effect from time t to time $t+1$

The generic recursive formula for policy value (annual case):

$$\left({}_tV + P_t - e_t\right)(1+i_t) = q_{[x]+t}\left(S_{t+1} + E_{t+1}\right) + p_{[x]+t}\ {}_{t+1}V$$

$$\text{Available assets at } t+1 = \text{Required assets at } t+1$$

Other versions

In addition to the prospective and recursive formulas we've seen for policy values, we can also derive various other formulas for the net policy value, usually by manipulating the prospective formula.

For example, consider a whole life policy with \$1 death benefit payable at the end of the year of death and annual premiums payable in advance, issued to (x).

$$
\begin{aligned}
{}_tV^n &= \underbrace{1 - d\ddot{a}_{x+t}}_{A_{x+t}} - \underbrace{\left(\frac{1}{\ddot{a}_x} - d\right)}_{P}\ddot{a}_{x+t} = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x} \\
{}_tV^n &= 1 - \frac{1 - A_{x+t}}{1 - A_x} = \frac{A_{x+t} - A_x}{A_x}
\end{aligned}
$$

Recursive formulae for continuos cash flows

Using the following notations:
$\bar{P}_t$: annual rate of premium payable at time t
$\bar{e}_t$: annual rate of premium related expenses payable at time t
S_t : death benefit payable if the insured dies at time t
E_t : termination related expenses at time t
δ_t : force of interest at time t
We can derive a continuous-time analog of our policy value recursion equation

Recursive formulae for continuos cash flows

Thiele's differential equation:

$$\frac{d}{dt}\ {}_tV = \delta_t\ {}_tV + \bar{P}_t - \bar{e}_t - \left(S_t + E_t - {}_tV\right)\mu_{[x]+t}$$

Compare with:

$$\left({}_tV + P_t - e_t\right)\left(1 + i_t\right) = q_{[x]+t}\left(S_{t+1} + E_{t+1}\right) + p_{[x]+t}\ {}_{t+1}V$$

Recursive formulae for continuos cash flows

Thiele's differential equation:

$$\frac{d}{dt}\ {}_tV = \delta_t\ {}_tV + \bar{P}_t - \bar{e}_t - (S_t + E_t - {}_tV)\,\mu_{[x]+t}$$

$\frac{d}{dt}\ {}_tV$: rate of the increase in the policy value at time t
$\delta_t\ {}_tV$: interest is earn on the current amount of policy value
$\bar{P}_t - \bar{e}_t$: premium income minus the premium related expenses is increasing the policy at rate $\bar{P}_t - \bar{e}_t$
$(S_t + E_t - {}_tV)\,\mu_{[x]+t}$: Claims plus claim related expenses decrease the amount of the policy value. The expected extra amount payable in the interval t to $t+h$ is $\mu_{[x]+t}\,(S_t + E_t - {}_tV)$

Recursive formulae for non-annual but discrete policies

Another complication we may encounter in valuation lies in dealing with policies having non-annual but discrete cash flows (i.e. monthly, quarterly, biannually).
We can use Thiele's Differential Equation as an approximation for a small time period h by assuming that:

$$\frac{d}{dt}\ {}_tV = \frac{1}{h}\left(\ {}_{t+h}V - \ {}_tV\right)$$

Hence,

$${}_{t+h}V - \ {}_tV = h\left\{\delta_t\ {}_tV + P_t - e_t - \left(S_t + E_t - \ {}_tV\right)\mu_{[x]+t}\right\}$$

This can be used recursively to find policy values at fractional ages.

Retrospective policy values

The basic formulas we've seen for policy values have been **prospective** in nature, meaning that at time t we're computing the policy value by considering what's expected to happen in the future.

We can also define a **retrospective** policy value by looking from time t back to the time of policy issue. The general form of a retrospective policy value is:

Retrospective policy value at time t = *accumulated value* at time t of *past premiums* - *accumulated value* at time t of *past benefits and expenses*

More on Retrospective policy values next week!