MTH6105 – Algorithmic Graph Theory **Spring 2024**

Problem Sheet 1 **F. Fischer**

You are expected to **attempt all exercises** before the seminar and to **actively participate** in the seminar itself.

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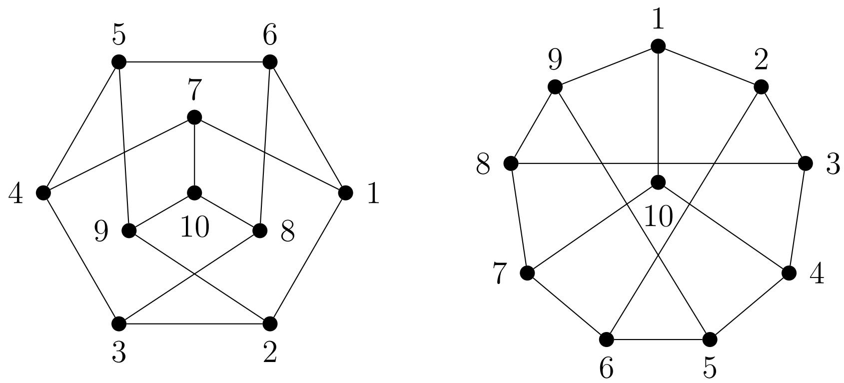
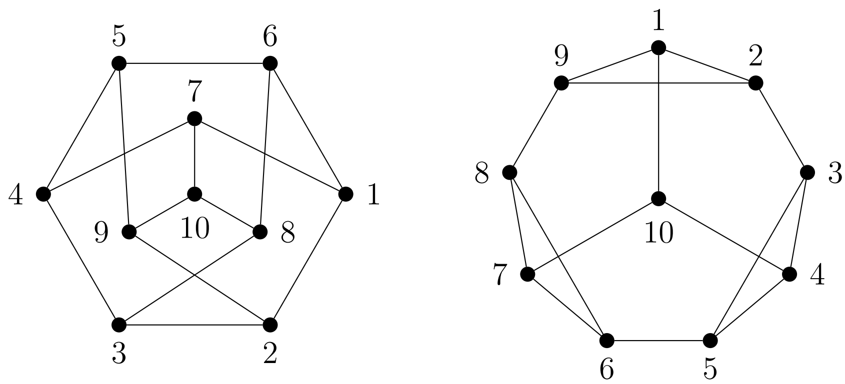
1. For each pair of graphs shown below, either show that they are isomorphic by giving a bijection or explain why they are not isomorphic.

(a)

(b)



(c)



Note: We have defined which properties a bijection must possess to show that two graphs are isomorphic, but have not discussed an algorithm for finding such a bijection. You may therefore have to look for a bijection by trial and error, and you will probably conclude that this is a bit tedious. To show that two graphs are *not* isomorphic, you have to argue that a bijection with the desired property does not exist. This is again likely to involve some trial and error.

Solution:

(a) The graphs are isomorphic. This is shown for example by the bijection ϕ from the set of vertices of the graph on the left to the set of vertices of the graph on the right with

$$\phi(1) = 1, \quad \phi(2) = 4, \quad \phi(3) = 2,$$
$$\phi(4) = 5, \quad \phi(5) = 3, \quad \phi(6) = 6.$$

(b) The graphs are isomorphic. This is shown for example by the bijection ϕ from the set of vertices of the graph on the left to the set of vertices of the graph on the right with

$$\phi(1) = 4, \quad \phi(6) = 3, \quad \phi(2) = 5, \quad \phi(7) = 10,$$
$$\phi(3) = 6, \quad \phi(8) = 2, \quad \phi(4) = 7, \quad \phi(9) = 9,$$
$$\phi(5) = 8, \quad \phi(10) = 1.$$

(c) The graphs are *not* isomorphic. This can be shown for example by observing that the graph on the right contains a cycle of length three, while the graph on the left does not.

Note that the bijections given for (a) and (b) are not the unique bijections to preserve adjacency, but that giving one such bijection suffices to show that two graphs are isomorphic.

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2. Define the complement of a simple graph G as the graph G^c with $V(G^c) = V(G)$ such that for all distinct $u, v \in V(G^c)$, $uv \in E(G^c)$ if and only if $uv \notin E(G)$. Consider a simple graph G with $|V(G)| \geq 6$.

   (a) Show that G or G^c must contain a vertex of degree at least 3.

   (b) Show that G or G^c must contain a subgraph that is isomorphic to C_3. To this end, you may want to consider the adjacencies among neighbors of a vertex of degree 3.

   (c) Show that in any group of at least 6 people, there are always 3 that mutually know each other, or 3 that mutually do not know each other.

Solution:

(a) Each of the $n - 1$ vertices in $V(G) \setminus \{v\}$ is a neighbor of v in either G or G^c, so by the pigeonhole principle v must have at least $\lceil (n-1)/2 \rceil \geq 3$ neighbors in either G or G^c.

(b) Let v be a vertex of degree at least 3 in G or G^c, which exists by Part a.

Assume first that v has degree at least 3 in G. If there exists an edge $xy \in E(G)$ with $x, y \in N_G(v)$, then $G[\{v, x, y\}]$ is isomorphic to C_3. If such an edge does not exist, then $G^c[N_G(v)]$ is isomorphic to K_m, where $m = |N_G(v)|$, and since $m \geq 3$ both $G^c[N_G(v)]$ and G^c must contain a subgraph isomorphic to C_3.

Now assume that v has degree at least 3 in G_c, and observe that the claim follows by the same argument if we exchange the roles of G and G_c.

(c) Consider representing people by the vertices of a graph G, and adding an edge between two vertices if the two people they represent know each other. Then a subgraph isomorphic to C_3, also called a triangle, in G corresponds to a set of 3 people that mutually know each other, whereas a triangle in G^c corresponds to a set of 3 people that mutually do not know each other. The claim thus follows from Part b.

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3. Show that isomorphism forms an equivalence relation on the set of simple graphs. To do this, you must show each of the following:

(a) (symmetry) For any simple graphs G and H, if G is isomorphic to H, then H is isomorphic to G.

(b) (reflexivity) For any simple graph G, G is isomorphic to G.

(c) (transitivity) For any simple graphs F, G, and H, if F is isomorphic to G and G is isomorphic to H, then F is isomorphic to H.

Solution:

(a) Assume that G is isomorphic to H. Then there exists a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. Since ϕ is a bijection, there is an inverse bijection $\phi^{-1} : V(H) \to V(G)$ such that $\phi(\phi^{-1}(x)) = x$ for all $x \in V(H)$. Let $u', v' \in V(H)$. Then $u'v' \in E(H)$ if and only if $\phi(\phi^{-1}(u'))\phi(\phi^{-1}(v')) \in E(H)$, and since ϕ is an isomorphism this is the case if and only if $\phi^{-1}(u')\phi^{-1}(v') \in E(G)$. Thus ϕ^{-1} is an isomorphism from H to G, which shows that H is isomorphic to G.

(b) Consider the identity bijection ϕ such that $\phi(x) = x$ for all $x \in V(G)$. Then, clearly, $uv \in G$ if and only if $\phi(u)\phi(v) \in E(G)$. Thus ϕ is an isomorphism from G to G, which shows that G is isomorphic to G.

(c) Assume that F is isomorphic to G and G is isomorphic to H, and consider an isomorphism $\phi : V(F) \to V(G)$ from F to G and an isomorphism $\psi : V(G) \to V(H)$ from G and H. Consider the composition $\psi \circ \phi : V(F) \to V(H)$, and observe that it is a bijection. Consider $u, v \in V(F)$. Since ϕ is an isomorphism from F to G, $uv \in E(F)$ if and only if $\phi(u)\phi(v) \in E(G)$. Since ψ is an isomorphism from G to H, $\phi(u)\phi(v) \in E(G)$ if and only if $\psi(\phi(u))\psi(\phi(v)) \in E(H)$. This means that $uv \in E(F)$ if and only if $(\psi \circ \phi)(u)(\psi \circ \phi)(v) \in E(H)$. Thus $\psi \circ \phi$ is an isomorphism from F to H, which shows that F and H are isomorphic.
