MTH6105 – Algorithmic Graph Theory — Spring 2024

Assessed Coursework 1 — F. Fischer

This assessment consists of three exercises, which carry equal weight and together contribute **10% of your mark for the module**. Please **upload your answers before the deadline**.

Any work you submit must be your own. You may discuss the exercises with other students, but you must write up your solution yourself. Copying a solution or submitting someone else's solution constitutes an assessment offence.

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1. Let the *degree sequence* of a graph G be the sequence of length $|V(G)|$ that contains the degrees of the vertices of G in non-increasing order.

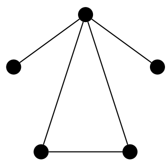
   (a) For each of the following sequences, either draw a simple graph whose degree sequence is equal to that sequence, or explain why such a graph does not exist: (i) $(4, 4, 4, 2, 2)$, (ii) $(4, 2, 2, 1, 1)$, (iii) $(3, 3, 3, 2, 1)$, (iv) $(4, 3, 3, 2, 1)$, (v) $(2, 2, 2, 1, 1)$.

   (b) Consider a simple graph with 9 vertices, such that the degree of each vertex is either 5 or 6. Prove that there are at least 5 vertices of degree 6 or at least 6 vertices of degree 5.

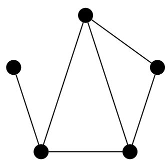
   Solution:

   (a) (i) A simple graph with degree sequence $(4, 4, 4, 2, 2)$ does not exist: each of the three vertices with degree 4 would have to be adjacent to all other vertices, which contradicts the existence of a vertex with degree 2.

   (ii) The following simple graph has the degree sequence $(4, 2, 2, 1, 1)$.

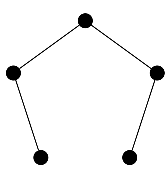


   (iii) The following simple graph has the degree sequence $(3, 3, 3, 2, 1)$.



   (iv) A simple graph with degree sequence $(4, 3, 3, 2, 1)$ does not exist, as by the handshake lemma the number of vertices with odd degrees must be even.

   (v) The following simple graph has the degree sequence $(2, 2, 2, 1, 1)$.



(b) Assume for contradiction that the claim is false, i.e., that there are at most 4 vertices of degree 6 and at most 5 vertices of degree 5. Since there are 9 vertices, this means that there are exactly 4 vertices of degree 6 and exactly 5 vertices of degree 5. This contradicts the fact that, by the handshake lemma, the number of vertices with odd degrees must be even.

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2. Let G be a graph and $e \in E(G)$. Let H be the graph with $V(H) = V(G)$ and $E(H) = E(G) \setminus \{e\}$. Then e is a *bridge* of G if H has a greater number of connected components than G.

   (a) Let G be the simple graph with $V(G) = \{u, v, w, x, y, z\}$ and $E(G) = \{uy, vx, vz, wx, xz\}$. For each $e \in E(G)$, state whether e is a bridge of G. Justify your answer.

   (b) Assume that G is connected and that e is a bridge of G with endpoints u and v. Show that H has exactly two connected components H_1 and H_2 with $u \in V(H_1)$ and $v \in V(H_2)$. To this end, you may want to consider an arbitrary vertex $w \in V(G)$ and use a u–w-path in G to construct a u–w-path or a v–w-path in H.

   (c) Show that e is a bridge of G if and only if it is *not* contained in a cycle of G.

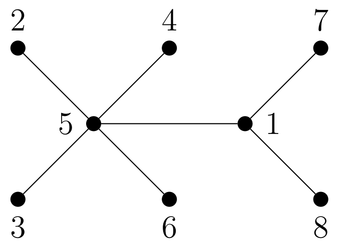
Solution:

(a) If an edge e is the unique edge incident to one of its endpoints, then e is a bridge: e forms a path so the two endpoints are in the same connected component; after removal of e one of its former endpoints is no longer the endpoint of any edge and forms a connected component on its own, while the connected components not containing the endpoints are unaffected by the removal of e. Thus uy is a bridge because it is the unique edge incident to u, and wx is a bridge because it is the unique edge incident to w. The remaining edges are contained in the cycle x, v, z, x, so by Part c none of them is a bridge.

(b) Since G is connected and e is a bridge, H must have at least two connected components. We show that there are exactly two connected components by showing that for every $w \in V(H)$ there is a u–w-path or a v–w-path in H. Let $w \in V(H)$. Since G is connected, G contains a u–w-path $v_0 e_1 v_1 \ldots e_m v_m$. If the path does not contain e, then it is a u–w-path in H. This means that w is contained in the same connected component of H as u. If the path does contain e, it must be the case that $e = e_1$. Then $v_1 \ldots e_m v_m$ is a v–w-path in H, which means that w is contained in the same connected component of H as v. We have shown that every vertex w is contained in the same connected component as either u or v. Since H has at least two connected components, it must thus have exactly two connected components H_1 and H_2 with $u \in V(H_1)$ and $v \in V(H_2)$.

(c) Let u and v be the endpoints of e. We may assume that G is connected, since otherwise we may just consider the connected component of G containing e. By Part b, e is a bridge of G if and only if there is no u–v-path in H. On the other hand, e is contained in a cycle of G if and only if removal of e from this cycle yields a u–v-path in H.

3. (a) Determine the Prüfer code of the following tree.



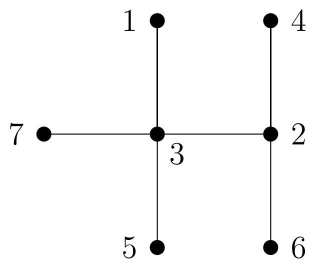
(b) Draw the trees with Prüfer codes (i) $(3, 2, 3, 2, 3)$, (ii) $(5, 5, 5, 5, 5)$, and (iii) $(6, 5, 4, 3, 2)$.

Solution:

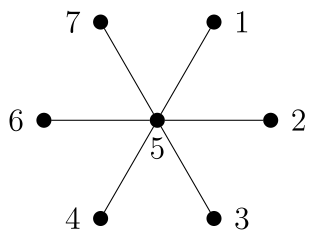
(a) The tree has Prüfer code $(5, 5, 5, 5, 1, 1)$.

(b) The trees are as follows:

(i)



(ii)



(iii)