Random Processes – 2023/24 Problem Sheet 1

The questions on this sheet are based on the material on the Markov property and transition probabilities from Week 1 lectures. I've also given pointers to some recent exam paper questions on this material[1]. This sheet is not for assessment. We will discuss selected questions from it in the Week 2 seminar.

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1. A standard fair 6-sided die is rolled repeatedly. Let X_i be the largest number seen in the first i rolls.

(a) Explain briefly why $(X_1, X_2, X_3, \ldots)$ is a Markov chain.

(b) Write down the state space, transition matrix and transition graph.

(c) I repeat the experiment using the fair die for the first 3 rolls and then switching to a biased die. What is different about this process?

A magician modifies the die so that it behaves normally on the first roll but thereafter it can never produce the same number twice in succession with the other 5 possibilities being equally likely. As before, X_i is the largest number seen in the first i rolls.

(d) Is $(X_1, X_2, X_3, \ldots)$ still a Markov chain? If your answer is yes give the new transition probabilities. If your answer is no give a specific violation of the Markov property.

The magician modifies the die again so that it behaves normally on the first roll but thereafter it can never produce a roll of k followed immediately by a roll of $6 - k$ with the other 5 possibilities being equally likely. For example if the first roll is 2 then the second roll cannot be 4. As before, X_i is the largest number seen in the first i rolls.

(e) Is $(X_1, X_2, X_3, \ldots)$ still a Markov chain? If your answer is yes give the new transition probabilities. If your answer is no give a specific violation of the Markov property.

[1] Past exam papers are available at the QMplus exam paper repository https://qmplus.qmul.ac.uk/mod/data/view.php?id=401329

2. Let $(X_0, X_1, X_2, \ldots)$ be a Markov chain on state space $\{1, 2, 3, 4\}$ with transition matrix

$$\begin{pmatrix} 0 & 1/4 & 3/4 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Draw the transition graph of the chain.

(b) Find the following probabilities, giving your reasoning in each case.

(i) $\mathbb{P}(X_1 = 3 \mid X_0 = 2)$
(ii) $\mathbb{P}(X_2 = 3 \mid X_1 = 2)$
(iii) $\mathbb{P}(X_2 = 3 \mid X_1 = 2, X_0 = 1)$
(iv) $\mathbb{P}(X_2 = 3, X_1 = 2 \mid X_0 = 1)$
(v) $\mathbb{P}(X_2 = 3 \mid X_0 = 3)$
(vi) $\mathbb{P}(X_{1000} = 1 \mid X_0 = 4)$

3. Let $(X_0, X_1, X_2, \ldots)$ be a Markov chain on state space $\{1, 2, 3, 4\}$ with the non-zero transition probabilities being

$$p_{11} = p_{44} = 1, \quad p_{21} = p_{23} = p_{24} = 1/3, \quad p_{31} = p_{32} = 1/2.$$

(a) Draw the transition graph.

(b) What is special about states 1 and 4?

(c) Calculate the following:

(i) $\mathbb{P}(X_n \neq 1, 4 \mid X_0 = 2)$,
(ii) $\mathbb{P}(X_n = 4 \mid X_0 = 2)$.

(d) What does (c) tell you about the behaviour of the process as n tends to infinity?

Some recent exam questions on the material in Week 1 include:

- January 2022 Exam. Question 4(a)
- January 2023 Exam. Question 4(a)

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