WEEK 1 NOTES

1. Basic concepts

1.1. **What is PDE.** A *partial differential equation* (PDE) is an equation for a function $U = U(x_1, \dots, x_n)$ of $n \geq 2$ variables involving partial derivatives of U. If the equation depends on only one variable one speaks of an *ordinary differential equation*. Partial differential equations are key to describing the fundamental interactions of Nature and in the modelling of a wide range of systems (economics, finance, population dynamics, ecology, ...).

Notation. *In this course we will systematically use the shorthand notation*

$$U_{x_i} \equiv \frac{\partial U}{\partial x_i}, \qquad U_{x_i x_j} \equiv \frac{\partial^2 U}{\partial x_i \partial x_j}, \qquad \cdots$$

Definition 1.1. Let $U(x_1, \dots, x_n)$ be a function of n variables. A PDE about the function U is an equation of the form

$$F(x_i, U, U_{x_i}, U_{x_i x_j}, \dots) = 0, i = 1, ..., n \tag{1.1}$$

Here F is a function about the variables $x_i's$, the unknown function U and its partial derivatives $U_x, U_y, \cdots$.

Note. *In this course we will be mostly interested in the case $n = 2$ so that $(x_1, x_2) = (x, y)$ or $(x_1, y_1) = (x, t)$ — the latter choice used in problems involving time.*

Concrete examples of pde's to be considered in this course are

(1.2a) $\quad U_x \pm U_t = 0 \quad$ (advection equation in $1 + 1$ dimensions),

(1.2b) $\quad U_{tt} - U_{xx} = 0 \quad$ (wave equation in $1 + 1$ dimensions),

(1.2c) $\quad U_{xx} + U_{yy} = 0 \quad$ (Laplace equation in 2 dimensions),

(1.2d) $\quad U_t - U_{xx} = 0 \quad$ (heat equation in $1 + 1$ dimensions).

Definition 1.2. The order of a pde is the highest derivative which appears in the equations.

Note. *In this course we will only consider equations of first and second order.*

The above 4 equations (and their variants) are the 4 main types of equation we will focus on solving in this module. They come from the mathematical modelling of some important physical phenomena.

Example 1.3 (Deduction of Heat equation in $1 + 1$ dimension)**.** .

Let $U(x, t)$ be the temperature at the point x at time t. Consider an infinite rod, which can be represented by $\mathbb{R}$. For any point x on the rod, focus on a small interval $I = [x - \frac{\delta}{2}, x + \frac{\delta}{2}]$, of length δ and centered at x.

The total heat in the interval I is

$$\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} U(y, t) dy,$$

and the change of total heat in I is

$$\frac{\partial}{\partial t}\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} U(y,t)dy = \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} \frac{\partial}{\partial t}U(y,t)dy$$

On the other hand, we can apply Newton's law of cooling, which states that heat flows from the higher to lower temperature at a rate proportional to the difference, that is, the gradient. The change rate of heat at the right end point is $CU_x(x+\frac{\delta}{2})$ and the change rate of heat at the left end point is $CU_x(x-\frac{\delta}{2})$. Here C is the heat constant of the material.

So the change of total heat can also be computed by

$$\frac{\partial}{\partial t}\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} U(y,t)dy = CU_x(x+\frac{\delta}{2},t) - CU_x(x-\frac{\delta}{2},t)$$

Divide both sides by δ and take the limit as $\delta \to 0$, we get

$$\begin{aligned}\lim_{\delta\to 0}\frac{\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} \frac{\partial}{\partial t}U(y,t)dy}{\delta} &= \lim_{\delta\to 0}\frac{CU_x(x+\frac{\delta}{2},t) - CU_x(x-\frac{\delta}{2},t)}{\delta}\\ U_t(x,t) &= CU_{xx}(x,t).\end{aligned}$$

Now define $\tilde{U}(x,t) = U(x,\frac{t}{C})$ by a change of variable. We then have $\tilde{U}$ satisfies the equation

$$\tilde{U} = \tilde{U}_{xx},$$

which is the heat equation in $1+1$ dimension.

1.2. **Linear PDEs and homogeneous PDEs.**

Definition 1.4. An operator is *linear* if

(i) $\mathcal{L}(U+V) = \mathcal{L}U + \mathcal{L}V$,
(ii) $\mathcal{L}(\alpha U) = \alpha\mathcal{L}U$

for any functions U, V and constant α.

A partial differential equation $\mathcal{L}U = f(x,y)$ is called *linear* whenever $\mathcal{L}$ is linear. Alternatively, a pde is linear if it is linear in U, U_x, U_y, $U_{xx}, \ldots$. If the equation is not linear, we say it is *non-linear*.

Example 1.5. Equations (1.2a), (1.2b), (1.2c) and (1.2d) are linear.

Example 1.6. The equation

$$U_{tt} - U_{xx} + U^2 = 0$$

is non-linear.

Example 1.7. The equation

$$U_{tt} - U_{xx} = \sin^2 U$$

is non-linear.

A concept which will be important in our discussion is the following:

Definition 1.8. Given a pde operator $\mathcal{L}$, an equation of the form

$$\mathcal{L}U = 0$$

is said to be *homogeneous*. An equation of the form

$$\mathcal{L}U = f,$$

with $f \neq 0$ a function is called *inhomogeneous*.

Example 1.9. The equation

$$U_{xx} + U_{yy} = 2$$

is linear but inhomogenous.

Notation. *For a 2-variable function $u = u(x,y)$, we will denote by $\Delta u = u_{xx} + u_{yy}$ for simplicity. This is a linear operator called the Laplace operator or Laplacian. The symbol Δ is pronounced as "Delta".*

1.3. **The principle of superposition.** Some important observations which will be used repeatedly are the following:

- If $U_1, U_2, \dots, U_N$ are solutions to $\mathcal{L}U = 0$, a linear pde, then
$$U_1 + \cdots + U_N$$
is also a solution. This observation is called the *principle of superposition* and is a key property of linear pde's. More about this later!
- If U solves the homogeneous linear equation $\mathcal{L}U = 0$ and V solves the inhomogeneous linear equation $\mathcal{L}V = g$ then $U + V$ solves the inhomogeneous equation. This can be seen from
$$\mathcal{L}(U+V) = \mathcal{L}U + \mathcal{L}V = 0 + g = g.$$

Example 1.10. Some solutions to the homogeneous and inhomogenelus Laplace equations in $\mathbb{R}^2$:

- $U_1(x,y) = x^2$ is a solution to the inhomogeneous PDE $\Delta U = 2$
- $U_2(x,y) = x + y$ is a solution to the homogeneous PDE $\Delta U = 0$
- $U_3(x,y) = U_1(x,y) + U_2(x,y) = x^2 + x + y$ is a solution to the inhomogeneous PDE $\Delta U = 2$

Sometimes we have to specify in which domain does a solution solve a PDE (because it may not holds for all the plane $\mathbb{R}^2$):

Example 1.11. $U_4(x,y) = \ln\sqrt{x^2+y^2}$ solves the Laplace equation $\Delta U = 0$ in $\mathbb{R}^2 \setminus \{(0,0)\}$. ($U_4$ is not defined for $(x,y) = (0,0)$!)

Example 1.12. $U(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is a solution to the heat equation $U_t = U_{xx}$.

To see this, we compute the 2nd partial derivatives with respect to x (using chain rule and product rule)

$$\begin{aligned} U_{xx}(x,t) &= \left[\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \cdot \frac{-x}{2t}\right]_x \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \cdot \frac{x^2}{4t^2} + \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \cdot \frac{-1}{2t}, \end{aligned}$$

and the partial derivative with respect to t

$$U_t(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \cdot \frac{x^2}{4t^2} + \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4t}} \cdot \frac{-}{2} \frac{1}{t^{\frac{3}{2}}}.$$

We check that $\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \cdot \frac{-1}{2t} = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4t}} \cdot \frac{-}{2} \frac{1}{t^{\frac{3}{2}}}$ and thus

$$U_{xx} = U_t$$

A final example that we give for solutions to the heat equation is constructed out of known solutions by rescaling.

Example 1.13. If $U(x,t)$ solves the equation $U_t = U_{xx}$, then so does $\tilde{U}(x,t) = U(Cx, C^2t)$ for any $C \neq 0$. This can be seen as follows:

By chain rule

$$\tilde{U}_t(x,t) = C^2 U_t(Cx, C^2t)$$
$$\tilde{U}_{xx}(x,t) = C^2 U_x(Cx, C^2t),$$

And thus if $U_t = U_{xx}$, we must also have

$$\tilde{U}_t = \tilde{U}_{xx}.$$

2. Solving some basic pde's

Start by looking at a very basic example, an ordinary differential equation (ode).

Example 2.1. Consider the ordinary differential equation for the function $U = U(t)$

$$\frac{dU}{dt} = 0.$$

The solution is given by

$$U(t) = c$$

with c a constant.

Consider now a function $U = U(x,y)$ of two variables.

Example 2.2. The solution of the pde

$$U_x = \frac{\partial U}{\partial x} = 0$$

is given (by integrating with respect to x) by

$$U(x,y) = f(y)$$

where f is a function of y only.

Note. *Whereas ode's have general solutions involving arbitrary constants, pde's have general solutions involving arbitrary functions of some of the coordinates.*

Consider now an extension of the previous example:

Example 2.3. Let

$$U_{xx} = \frac{\partial^2 U}{\partial U^2} = 0.$$

Integrating once with respect to x one finds that

$$U_x = f(y)$$

as in the previous example. Integrating once more one finds

$$U(x,y) = xf(y) + g(y)$$

with f, g arbitrary functions of y.

A more sophisticated example is:

Example 2.4. Consider the equation
$$U_{xy} = 0.$$
Integrating once with respect to x one finds that
$$U_y = f(y).$$
Now, integrating with respect to y one has
$$U(x, y) = g(x) + \int f(y)dy.$$
where g is a function of x only. But $\int f(y)dy$ is, in fact, a function of y so we can actually write
$$U(x, y) = g(x) + F(y)$$
with $F(x) \equiv \int f(y)dy$. We can readily check that the above is, indeed, a solution by direct differentiation.

Note. *Recall that if a function $U(x, y)$ can be differentiated twice, then*
$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x},$$
or in terms of our new notation
$$U_{xy} = U_{yx}.$$

Example 2.5. Consider the equation
$$U_{xx} + U = 0.$$
It can be checked that the solution is given by
$$U(x, y) = f(y)\cos x + g(y)\sin x.$$
The above equation should be compared with the ode
$$z'' + z = 0.$$

Note. *The above example shows that often it is useful to pretend that $U(x, y) = U(x)$ and then see what ode arises.*

A similar example to the previous one is:

Example 2.6. Let
$$U_x = 2x\sin y + e^{xy}.$$
Direct integration gives
$$U(x, y) = x^2 \sin y + \frac{e^{xy}}{y} + f(y).$$

And finally two more examples which will be further elaborated during the course:

Example 2.7. One can readily verify by direct computation that
$$U(x, y) = \sin(nx)\sinh(ny)$$
solves
$$U_{xx} + U_{yy} = 0.$$

Example 2.8. If f is a differentiable function of one variable and $c \neq 0$ is a constant, then

$$U(x,t) = f(x-ct)$$

satisfies the advection equation

$$U_t + cU_x = 0.$$

For example, if $f(z) = \sin z$ then

$$f(x-ct) = \sin(x-ct).$$

The assertion can be verified using the chain rule for ordinary derivatives.

Note. *Recall that if $f = f(x)$ and $g = g(x)$ are two differentiable functions of x then the derivative of the composition $f \circ g$ is given by*

$$\frac{df \circ g}{dx} = \frac{d}{dx} f(g(x)) = \frac{df(g(x))}{dg}\frac{dg}{dx}.$$

Let's see next that the solutions of the form $f(x-ct)$ actually make up all the possible solutions to this advection equation.

3. Solving first order linear PDEs

In this section we discuss how to obtain the solutions of the partial differential equation

$$aU_x + bU_y = 0, \tag{3.1}$$

with a, $b \neq 0$ some constants. This equation is a first order homogeneous equation. We will analyse two methods to obtain the solution to this equation.

Before introducing the first method, we recall the chain rule for partial derivatives.

3.1. **The chain rule for partial derivatives.**

An important tool in the analysis of pde's is the chain rule for partial derivatives. Given the usual coordinates (x,y) on $\mathbb{R}^2$ consider new coordinates $(\tilde{x},\tilde{y})$ given by an expression of the form

$$\tilde{x} = \tilde{x}(x,y), \qquad \tilde{y} = \tilde{y}(x,y).$$

That is, we assume that $(\tilde{x},\tilde{y})$ can be written as functions of the old coordinates (x,y). One is then interested in the relation between the partial derivatives $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial\tilde{x}$, $\partial/\partial\tilde{y}$. This is given by the *chain rule* for partial derivatives which, in the language of operators takes the form:

$$\frac{\partial}{\partial x} = \frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}} + \frac{\partial\tilde{y}}{\partial x}\frac{\partial}{\partial\tilde{y}},$$
$$\frac{\partial}{\partial y} = \frac{\partial\tilde{x}}{\partial y}\frac{\partial}{\partial\tilde{x}} + \frac{\partial\tilde{y}}{\partial y}\frac{\partial}{\partial\tilde{y}}.$$

Note. *Observe the pattern in the above seemingly complicated equations which helps to remember the formulae.*

3.2. **Method 1: Solution by change of coordinates (analytic approach).** A general observation which is often very useful is that a change of variables can turn a seemingly hard problem into an easy one. We try this approach here.

In what follows we consider the change of variables given by

$$\tilde{x}(x,y) = ax + by,$$
$$\tilde{y}(x,y) = bx - ay.$$

We now express equation (3.1) in terms of the coordinates $(\tilde{x},\tilde{y})$. For this, we make use of the chain rule. One has that

$$U_x = \frac{\partial U}{\partial x} = \frac{\partial \tilde{x}}{\partial x}\frac{\partial U}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial U}{\partial \tilde{y}} = aU_{\tilde{x}} + bU_{\tilde{y}},$$
$$U_y = \frac{\partial U}{\partial y} = \frac{\partial \tilde{x}}{\partial y}\frac{\partial U}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial y}\frac{\partial U}{\partial \tilde{y}} = bU_{\tilde{x}} - aU_{\tilde{y}}.$$

Substituting these expressions into the left hand side of equation (3.1) one has that

$$\begin{aligned} aU_x + bU_y &= a(aU_{\tilde{x}} + bU_{\tilde{y}}) + b(bU_{\tilde{x}} - aU_{\tilde{y}}) \\ &= (a^2+b^2)U_{\tilde{x}}. \end{aligned}$$

Thus, one concludes that in terms of the coordinates $(\tilde{x},\tilde{y})$, equation (3.1) takes the form

$$U_{\tilde{x}} = 0.$$

We already know how to solve this equation. Namely one has that

$$U(\tilde{x},\tilde{y}) = f(\tilde{y}),$$

where f is a function only of the coordinate $\tilde{y}$. We can rewrite this expression in terms of the coordinates (x,y) as

$$U(x,y) = f(bx - ay). \tag{3.2}$$

That is, $U(x,y)$ depends only on the combination $bx-ay$. The formula (3.2) is the *general solution* of equation (3.1). Observe that it involves an arbitrary function.

Before we introduced the second method of solving the 1st order linear PDE, we need to review some notions from calculus.

3.3. **Gradient and directional derivatives.** Given a function $f = f(x,y)$ the *gradient* ∇f is the vector defined by

$$\nabla f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (f_x, f_y).$$

Geometrically, $f = f(x,y)$ can be thought of as a surface in $\mathbb{R}^3$ where the z coordinate is given by the function f. At a given point (x,y), the gradient gives the direction of maximum growth (*steepest slope*) of f.

Now, given a vector $\vec{v} = (v_1, v_2)$ on $\mathbb{R}^2$, the *directional derivative* $\nabla_{\vec{v}} f$ of the function $f = f(x,y)$ in the direction of f is defined by

$$\nabla_{\vec{v}} f \equiv \vec{v}\cdot\nabla f = v_1 f_x + v_2 f_y,$$

where $\cdot$ denotes the *inner product (dot product)*. This derivative gives the change of f in the direction of $\vec{v}$.

3.4. **Method 2: Geometric approach.** By taking a *geometric approach*, one can understand where do the change of variables we used comes from.

The basic observation is the following:

$$\begin{aligned} aU_x + bU_y &= (a,b)\cdot(U_x,U_y) \\ &= (a,b)\cdot\nabla U \\ &= \nabla_{\vec{v}}U, \end{aligned}$$

where $\vec{v} \equiv (a,b)$. Thus, equation (3.1) means geometrically that the function U is constant in the direction of $\vec{v}$.

Question 3.1. What curves have tangent given by the *constant* vector $\vec{v} = (a,b)$?

The curves necessarily have to be lines! The lines have slope $dy/dx = b/a$ so that their equation is of the form

$$y = \frac{b}{a}x + c, \qquad c \quad \text{a constant.}$$

The last expression can be rewritten as

$$bx - ay = c. \tag{3.3}$$

From the previous discussion it follows that the solution is constant along these lines —we call these lines *characteristic lines*. Thus, the function $U(x,y)$ depends on the value of c only and one can write

$$U(x,y) = f(c) = f(bx - ay)$$

Observe that the result we have obtained coincides with what we had using the method 1 (analytic approach).