Queen Mary
University of London

Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH5126: Statistics for Insurance

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

IFoA exemptions. For actuarial students, this module counts towards IFoA actuarial exemptions. You are allowed two submissions for this exam—the first for your IFoA mark, and the second for your module mark. To be eligible for IFoA exemptions, **your IFoA submission must be within the first 3 hours of the assessment period**.

Examiners: G. Ng, S. Liverani

Question 1 [21 marks]. Artemis Insurance plc considers that claims for its motor insurance portfolio occur in accordance with a compound Poisson process. The claim frequency for the whole portfolio is 100 per annum and individual claims, X, have an exponential distribution with a mean of £8,000.

(a) Derive the moment generating function $M_X(t)$. **[6]**

(b) Calculate the adjustment coefficient, R, if the total premium rate for the portfolio is £1,000,000 per annum. **[6]**

(c) Verify that the calculated value of R satisfies the inequality:

$$R < \frac{2[\frac{c}{\lambda} - E(X)]}{E(X^2)}$$

[4]

(d) Find to the nearest £, the minimum surplus required to ensure the probability of ultimate ruin is less than p.
If the last digit of your student ID is 0, use $p = 10\%$.
If the last digit of your student ID is 1, use $p = 11\%$.
If the last digit of your student ID is 2, use $p = 12\%$.
If the last digit of your student ID is 3, use $p = 13\%$.
If the last digit of your student ID is 4, use $p = 14\%$.
If the last digit of your student ID is 5, use $p = 15\%$.
If the last digit of your student ID is 6, use $p = 16\%$.
If the last digit of your student ID is 7, use $p = 17\%$.
If the last digit of your student ID is 8, use $p = 18\%$.
If the last digit of your student ID is 9, use $p = 19\%$.

[5]

Solution

(a)

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \int_0^\infty e^{tx} \frac{1}{8,000} e^{-\frac{1}{8,000}x} dx$$

$$= \frac{1}{8,000} \int_0^\infty e^{-(\frac{1}{8,000}-t)x} dx$$

$$= \frac{\frac{1}{8,000}}{-(\frac{1}{8,000}-t)} \left[e^{-(\frac{1}{8,000}-t)x} \right]_0^\infty$$

(provided that $t < \frac{1}{8,000}$)

$$= \frac{1}{1-8,000t}$$

(provided that $t < \frac{1}{8,000}$)

[6]

(b) The adjustment coefficient is the unique positive root of:

$$M_X(R) = 1 + \frac{cR}{\lambda}$$

where c is the rate of premium income and λ is the Poisson parameter.

Since individual claims have an exponential distribution with mean of 8,000:

$$M_X(R) = E(e^{RX}) = \frac{1}{1-8,000R}, \quad t < \frac{1}{8,000}$$

So, the adjustment coefficient is the unique positive root of:

$$\frac{1}{1-8,000R} = 1 + \frac{1,000,000R}{100}$$

[3]

Rearranging:

$$(1+10,000R)(1-8,000R) = 1$$

$$\implies 1 + 2,000R - 80,000,000R^2 = 1$$

$$\implies 2,000R(1-40,000R) = 0$$

The adjustment coefficient is the unique positive root solution, i.e.:

$$R = \frac{1}{40,000} = 0.000025$$

Note: Student has to mention that R is defined to be the unique positive root.

[3]

(c) Since X has an exponential distribution:

$$E(X) = 8,000$$

and :

$$E(X^2) = var(X) + [E(X)]^2$$
$$= 8,000^2 + 8,000^2 = 2 \times 8,000^2$$

[**2**]

The inequality now reads:

$$R < \frac{2(1,000,000/100 - 8,000)}{2 \times 8,000^2} = 0.00003125$$

The exact value of R (0.000025) is indeed less than this. [**2**]

(d) The upper bound for the probability of ultimate ruin is given by the Lundberg's inequality as:

$$\psi(U) \leq e^{-RU} = e^{-0.000025U}$$

We want $e^{-0.000025U} < p$ [**2**]

Taking logs:

$$-0.000025U < \ln(p)$$

$$\implies U > \frac{\ln(p)}{-0.000025}$$
$$U > 92,103.404 \quad \text{for } p = 10\%$$

So to the nearest £, minimum surplus required is £92,104.

[**3**]

For p = 11%; 12%; 13%; 14%; 15%; 16%; 17%; 18%; 19%:
U > 88,290.997; 84,810.541; 81,608.833;
78,644.514; 75,884.799; 73,303.259;
70,878.274; 68,591.937; 66,429.248.

So to the nearest £, minimum surplus required is:
£88,291; £84,811; 81,609;
£78,645; £75,885; 73,304;
£70,879; £68,592; 66,429.

Question 2 [22 marks].
Petz, a general insurance company, is planning to set up a new class of pet insurance. Claims are expected to occur according to a Poisson process with parameter 60. Individual claims are thought to have a gamma distribution with parameters $\alpha = 150$ and $\beta = 0.5$.

It plans to raise funds of £1 million to start the business. A premium loading factor of 20% will be applied.

In the questions that follow, assume that the number of policies sold at outset remains the same.

(a) If Petz only managed to raise 90% of the funds it had expected to raise and went ahead to start the business, will the probability of ultimate ruin for Petz increase, decrease or remain unchanged? Explain your answer. **[3]**

(b) Calculate the mean and variance of the individual claims. **[2]**

(c) Preliminary claims data now indicate that individual claims have a gamma distribution with parameters $\alpha = 150$ and $\beta = 0.25$ instead. Calculate the new mean and variance of the individual claims. **[2]**

(d) Given the above change in the distribution of individual claims, will the probability of ultimate ruin for this business increase, decrease or remain unchanged? Explain your answer. **[3]**

(e) Preliminary claims data now indicate that the Poisson parameter is 80 instead. Will the premium received by Petz increase, decrease or remain unchanged? Explain your answer. **[3]**

(f) Will the above change in the Poisson parameter result in an earlier or later timing at which ruin may occur for Petz? Explain your answer. **[3]**

(g) How will the above change in the Poisson parameter affect the probability of ultimate ruin for Petz? Explain your answer. **[3]**

(h) If a 30% premium loading factor is applied instead, will the probability of ultimate ruin for Petz increase, decrease or remain unchanged? Explain your answer. **[3]**

Solution

(a) Here, there is an decrease in the insurer's initial surplus without any corresponding decrease in claim amounts.
So, there is an decrease in the insurer's security (less of a buffer to withstand claims).
Hence, the probability of ruin will increase.

[3]

(b) Mean is $\frac{150}{0.5} = 300$.
Variance is $\frac{150}{0.5^2} = 600$. **[2]**

(c) Mean is $\frac{150}{0.25} = 600$.
Variance is $\frac{150}{0.25^2} = 2,400$. [2]

(d) Therefore the claims are larger on average and more uncertain.
Both of these factors will increase the probability of ultimate ruin. [3]

(e) Cost of claims will increase.
The premium received will also increase proportionally.
This is because premium is (1 + premium loading factor) × cost of claims or $c = (1+\theta)\lambda m_1$. [3]

(f) The Poisson parameter has increased so claims occur more often.
Hence the timing at which ruin may occur will be earlier. [3]

(g) The Poisson parameter has increased so claims outgo and premiums received increase proportionally.
Also, claims occur more often, hence the timing at which ruin may occur will be earlier, but not the probability of it occurring in the first place.
Therefore the probability of ultimate ruin will remain unchanged.

[3]

(h) Since there is a higher loading factor, the premiums received will increase (even though claims and expenses) remain the same. [2]
Hence, the probability of ultimate ruin will decrease. [1]

Question 3 [9 marks]. Estimate the parameters to fit a $\text{Pareto}(\alpha, \lambda)$ distribution to the data in the table below by the method of moments.

Claim size (£)	**Average size** (£)	**Number of claims**
0-24	17.14	168
25-49	34.88	474
50-74	60.02	329
75-99	87.73	205
100-199	134.27	367
200-499	282.88	241
500-999	685.29	80
1000-4999	2455.83	84
5000 and over	7170.00	2

[9]

Solution

We need to calculate the sample mean, $\frac{\sum x}{n}$

$$\sum x = 17.14 \times 168 + 34.88 \times 474 + 60.02 \times 329 + 87.73 \times 205 + 134.27 \times 367 + 282.88 \times 241 + 685.29 \times 80 + 2455.83 \times 84 + 7170 \times 2 = 450,047.96$$

$n = 1,950$ so sample mean $= \frac{450,047.96}{1,950} = 230.7938$ [2]

and the sample variance, $\frac{\sum x^2}{n} -$ (sample mean $)^2$

$$\frac{\sum x^2}{n} = (17.14^2 \times 168 + 34.88^2 \times 474 + 60.02^2 \times 329 + 87.73^2 \times 205 + 134.27^2 \times 367 + 282.88^2 \times 241 + 685.29^2 \times 80 + 2455.83^2 \times 84 + 7170^2 \times 2)/1950 = 676,290,605.6948/1,950 = 346,815.6952$$

So sample variance =
$\frac{\sum x^2}{n} -$ (sample mean $)^2 = 346,815.6952 - 230.7938^2 = 293,549.9053$ [3]

Having calculated the sample mean and variance we can now compare the sample mean to the Pareto mean:

$$\frac{\lambda}{\alpha - 1} = 230.7938$$

[1]

and the sample variance to the Pareto variance:

$$\frac{\lambda^2\alpha}{(\alpha-1)^2(\alpha-2)} = 293,549.9053$$

[**1**]

Squaring the first equation and dividing into the second we get:

$$\frac{\alpha}{\alpha-2} = 5.51104012$$
$$\hat{\alpha} = 2.44335673$$
$$\frac{\lambda}{\alpha-1} = 230.7938 \implies \hat{\lambda} = 230.7938 \times (2.44335673 - 1) = 333.117822$$

So $\hat{\alpha} = 2.44335673$ and $\hat{\lambda} = 333.117822$

[**2**]

Question 4 [12 marks]. A general insurance company has a portfolio of policies under which individual loss amounts follow an exponential distribution with mean $\frac{1}{\lambda}$. An excess of loss reinsurance arrangement is in place with retention level 120.

In one year, the insurer observes:

75 claims for amounts below 120 with mean claim amount 55, and
33 claims for amounts above the retention level.

Find the maximum likelihood estimate of λ.

[**12**]

Solution
The likelihood function corresponding to the 75 claims is:

$$\prod_{i=1}^{75} \lambda e^{-\lambda x_i} = \lambda^{75} e^{-\lambda \sum_{i=1}^{75} x_i}$$
$$= \lambda^{75} e^{-\lambda \times 75 \times 55}$$

(since we are given that sample mean, $\frac{\sum_{i=1}^{75} x_i}{75} = 55$) [**3**]
The probability of a claim exceeding 120 is

$$\int_{120}^{\infty} \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}\right]_{120}^{\infty} = e^{-120\lambda}$$

[**2**]

The complete likelihood function is :

$$L(\lambda) = \lambda^{75} e^{-\lambda \times 75 \times 55} (e^{-120\lambda})^{33}$$
$$= \lambda^{75} e^{-8085\lambda}$$

[**2**]

So the log-likelihood is:

$$\ln L = 75 \ln \lambda - 8085\lambda$$

[**1**]

Differentiating with respect to λ :

$$\frac{\partial}{\partial\lambda}\ln L = \frac{75}{\lambda} - 8085$$

Equating this to zero and solving, we then find that:

$$\hat{\lambda} = \frac{75}{8085} = 0.009276$$

[**3**]

We should check that we do have a maximum:

$$\frac{\partial^2}{\partial\lambda^2}\ln L = -\frac{75}{\lambda^2} < 0$$

so we have a maximum. [**1**]

Question 5 [26 marks]. The aggregate claims from a risk have a compound Poisson distribution with parameter μ. Individual claim amounts have a Pareto distribution with parameters $\alpha = 3$ and $\lambda = 1,000$.

The insurer of this risk calculates the premium using a premium loading factor of 0.25 (this means they charge 25% in excess of the risk premium).

The insurer is considering effecting excess of loss reinsurance with a retention limit of 1,000. The reinsurance premium would be calculated using a premium loading factor of 0.25.

The insurer's profit is defined to be the premium charged by the insurer less the reinsurance premium and less the claims paid by the insurer, net of reinsurance.

(a) Calculate the variance of the insurer's aggregate claims before reinsurance. [**6**]

(b) Calculate the variance of the insurer's aggregate claims net of reinsurance. [**12**]

(c) Hence, calculate the percentage change in the standard deviation of the insurer's profit as a result of effecting the reinsurance. [**8**]

Solution

(a) Aggregate claims before reinsurance, S, follows a compound Poisson distribution with Poisson parameter μ.

$$S = X_1 + X_2 + \cdots + X_N$$

where X ~Pareto(3,1000).

[**1**]

So using results for the variance of a compound Poisson distribution:

$$var(S) = \mu E(X^2)$$

$$= \mu(var(X) + [E(X)]^2)$$

[**2**]

Since X ~Pareto(3,1000),

$$E(X) = \frac{\lambda}{\alpha - 1} = 500$$

$$var(X) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = 750,000$$

[**2**]

So var(S) $= \mu(750,000 + 500^2) = 1,000,000\mu$

[**1**]

(b) Aggregate claims net of reinsurance, S_I, also follows a compound Poisson distribution with Poisson parameter μ.

$$S_I = Y_1 + Y_2 + \cdots + Y_N$$

where

$$Y = \begin{cases} X & \text{if } X \leq 1{,}000 \\ 1{,}000 & \text{if } X > 1{,}000 \end{cases}$$

[2]

So using results for the variance of a compound Poisson distribution:

$$var(S_I) = \mu E(Y^2)$$

[1]

$$E(Y^2) = \int_0^{1{,}000} x^2 \frac{3 \times 1{,}000^3}{(1{,}000 + x)^4} dx + \int_{1{,}000}^{\infty} 1{,}000^2 \frac{3 \times 1{,}000^3}{(1{,}000 + x)^4} dx$$

[2]

$$= 3 \times 1{,}000^3 \int_0^{1{,}000} \frac{x^2}{(1{,}000 + x)^4} dx + 3 \times 1{,}000^5 \int_{1{,}000}^{\infty} \frac{1}{(1{,}000 + x)^4} dx$$

The second integral is just:

$$\left[\frac{(1000 + x)^{-3}}{-3} \right]_{1000}^{\infty} = \frac{1}{3 \times 2{,}000^3}$$

[2]

For the first integral, put $u = 1{,}000 + x$ to obtain:

$$\int_{1{,}000}^{2{,}000} \frac{(u - 1{,}000)^2}{u^4} du = \left[-\frac{1}{u} + \frac{1{,}000}{u^2} - \frac{1{,}000{,}000}{3u^3} \right]_{1{,}000}^{2{,}000} = \frac{1}{24{,}000}$$

[3]

So:

$$E(Y^2) = \frac{3 \times 1{,}000^3}{24{,}000} + \frac{3 \times 1{,}000^5}{3 \times 2{,}000^3} = 250{,}000$$

So $var(S_I) = 250{,}000\mu$

[2]

(c) Consider the variance of the profit, firstly without reinsurance. The insurer's profit is equal to premiums charged less claims paid, S. Since only the claims are random (premiums are constants), the variance of the profit before reinsurance is the same as the variance of the aggregate claims, which we worked out in part (a). **[3]**

So the standard deviation of the profit before reinsurance is $1{,}000\sqrt{\mu}$. **[1]**

With reinsurance, the insurer's profit is equal to premiums charged less the reinsurance premium less the net claims paid, S_I. Since only the claims are random (premiums are constants), the variance of the profit is equal to the variance of S_I, which we worked out in part (b). [**2**]

So the standard deviation of the profit after reinsurance is $500\sqrt{\mu}$. [**1**]

So the percentage change in standard deviation of profit is a reduction of 50%. [**1**]

Question 6 [10 marks]. Individual claims from a portfolio are believed to have a $\mathsf{Pareto}(\alpha, \lambda)$ distribution. In Year 0, $\alpha = 6$ and $\lambda = 1000$.

Inflation is at a constant rate of r per annnum.

If the last digit of your student ID is 0, use $r = 10\%$.
If the last digit of your student ID is 1, use $r = 11\%$.
If the last digit of your student ID is 2, use $r = 12\%$.
If the last digit of your student ID is 3, use $r = 13\%$.
If the last digit of your student ID is 4, use $r = 14\%$.
If the last digit of your student ID is 5, use $r = 15\%$.
If the last digit of your student ID is 6, use $r = 16\%$.
If the last digit of your student ID is 7, use $r = 17\%$.
If the last digit of your student ID is 8, use $r = 18\%$.
If the last digit of your student ID is 9, use $r = 19\%$.

(i) Find the distribution of the individual claim payments in Year 1. [**8**]

(ii) Hence, state the distribution of the individual claim payments in Year 2. [**2**]

Solution

(i) Let X_0 represent the size of a Year 0 claim.

Then the distribution of X_0 is $\mathsf{Pareto}(6, 1000)$ and $X_1 = kX_0$ is the size of a Year 1 claim, where $k = 1 + r$.

To find the distribution of X_1, we put $u = kx$ in the integral of the PDF:

$$1 = \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}} dx$$

[2]

$$= \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda + \frac{u}{k})^{\alpha+1}} d\frac{u}{k}$$

[2]

$$= \int_0^\infty \frac{\alpha(k\lambda)^\alpha}{(k\lambda + u)^{\alpha+1}} du$$

[2]

Observe that the integrand is now the pdf of a $\mathsf{Pareto}(\alpha, k\lambda)$ distribution.
So the distribution of X_1 is $\mathsf{Pareto}(\alpha, k\lambda)$.
For $r = 10\%$, the claim distribution in Year 1 is $\mathsf{Pareto}(6, 1100)$.

[2]

For $r = 11\%$; 12%; 13%; 14%; 15%; 16%; 17%; 18%; 19%:
Claims distribution in Year 1:
$\mathsf{Pareto}(6, 1110)$; $\mathsf{Pareto}(6, 1120)$; $\mathsf{Pareto}(6, 1130)$;
$\mathsf{Pareto}(6, 1140)$; $\mathsf{Pareto}(6, 1150)$; $\mathsf{Pareto}(6, 1160)$;
$\mathsf{Pareto}(6, 1170)$; $\mathsf{Pareto}(6, 1180)$; $\mathsf{Pareto}(6, 1190)$.

(ii) Similarly, the distribution of X_2 is $\mathsf{Pareto}(\alpha, k^2\lambda)$.
For $r = 10\%$, the claim distribution in Year 2 is $\mathsf{Pareto}(6, 1210)$.

[2]

For $r = 11\%$; 12%; 13%; 14%; 15%; 16%; 17%; 18%; 19%:
Claims distribution in Year 2:
$\mathsf{Pareto}(6, 1232.1)$; $\mathsf{Pareto}(6, 1254.4)$; $\mathsf{Pareto}(6, 1276.9)$;
$\mathsf{Pareto}(6, 1299.6)$; $\mathsf{Pareto}(6, 1322.5)$; $\mathsf{Pareto}(6, 1345.6)$;
$\mathsf{Pareto}(6, 1368.9)$; $\mathsf{Pareto}(6, 1392.4)$; $\mathsf{Pareto}(6, 1416.1)$.

End of Paper.