Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH5126: Statistics for Insurance

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

For actuarial students only: This module also counts towards IFoA exemptions. For your submission to be eligible, **you must submit within the first 3 hours.**

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper** as notes for the exam.

Only approved non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: L. Fang, M. Nica

Question 1 [15 marks]. Let the random variable S denote the aggregate claims paid by the insurer, the random variable X_i denote the amount of the i-th claim, and $\{X_i\}_{i=1}^n$ are independent and identical random variables. $S = \sum_{i=1}^N X_i$, where the discrete random variable N represents the number of claims.
Suppose N follows a Poisson(λ) distribution, whose MGF is given by:

$$M_N(t) = e^{\lambda(e^t-1)}.$$

The mean and variance of the Poisson distribution are both λ.
All $\{X_i\}_{i=1}^n$ follow Normal(μ, σ^2), whose MGF is given by:

$$M_X(t) = e^{\mu t+\frac{1}{2}\sigma^2 t^2}.$$

You can use the following formulae without proof:
The moment generating functions (MGF) of S, X_i, N satisfy the equation
$M_S(t) = M_N[log M_X(t)]$.

(a) Find an expression for the MGF of the aggregate claim amount if the number of claims has a Poisson(100) distribution and individual claim sizes are Normal(20, 6^2). [**8**]
Solution:

$$M_X(t) = e^{20t+18t^2}$$
$$M_N(t) = e^{100(e^t-1)}$$
$$M_S(t) = M_N[log M_X(t)] = e^{100(e^{20t+18t^2}-1)}$$

(b) Find the mean and variance of the aggregate claim amount. [**7**]
Solution:

$$E(S) = \lambda m_1 = 200$$
$$m_2 = \sigma^2 + \mu^2 = 436$$
$$Var(S) = \lambda m_2 = 43600.$$

Parts a, b similar to the example in lecture.
IFoA CS2 syllabus area 1.2.4.

Question 2 [30 marks]. An insurer has effected excess of loss reinsurance with retention level 700. The annual aggregate claim amount from the insurer's risk has a compound Poisson distribution with Poisson parameter 20. Individual claim amounts are **uniformly** distributed on [0, 1000].

(a) Calculate the mean and variance of the **insurer's** aggregate claims under this reinsurance arrangement, S_I. **[10]**

Solutions:

Y_i is defined as the claim payments of insurers.

$$\begin{aligned} E(Y_i) &= \int_0^M xf(x)\,dx \ + MP(X_i > M) \\ &= \int_0^{700} 0.001x\,dx \ + 0.3M \\ &= 455. \end{aligned}$$

Therefore,

$$E(S_I) = \lambda E(Y_i) = 20 \times 455 = 9100.$$

$$\begin{aligned} E(Y_i^2) &= \int_0^M x^2 f(x)\,dx \ + M^2 P(X_i > M) \\ &= \int_0^{700} 0.001x^2\,dx \ + 0.3M^2 \\ &= 261,333. \end{aligned}$$

Therefore,

$$var(S_I) = \lambda E(Y_i^2) = 20E(Y_i^2) = 5,226,660$$

(b) Calculate the mean and variance of the **reinsurer's** aggregate claims under this reinsurance arrangement S_R. **[10]**

Solutions:

Z_i is defined as the claim payments of reinsurers.

$$\begin{aligned} E(S_R) &= E(S) - E(S_I) = \lambda E(Xi) - E(S_I) \\ &= 20 \times 500 - 9100 = 900 \end{aligned}$$

$$E(Z_i^2) = \int_M^{1000} (x-M)^2 f(x)\,dx \ = \int_0^{1000-M} y^2 0.001\,dy \ = 9000$$

Therefore,

$$var(S_R) = \lambda E(Z_i^2) = 20E(Y_i^2) = 180,000$$

(c) What is the variance of S, the aggregate claim amount **before reinsurance**? [**10**]

Solutions:

$$E(X_i^2) = \int_0^{1000} \frac{x^2}{1000}\, dx \; = \frac{1000000}{3}$$

Therefore,

$$var(S) = \lambda E(X_i^2) = 10 \times \frac{1000000}{3} = 3,333,333.33$$

Parts a, b, c application of lecture material.
IFoA CS2 syllabus areas 1.1.4, 1.2.4, 1.2.5.

Question 3 [30 marks].

(a) Explain, in words, the meaning of the following copula expression: $C(u, v)$. [**2**]

Solution:
This gives the probability that Random Variable 1 is in the bottom uth percentile, and Random Variabl 2 is in the bottom vth percentile.

(b) A **Clayton** copula is defined in the bivariate case as:

$$C[u, v] = (u^{-\alpha} + v^{-\alpha} - 1)^{-\frac{1}{\alpha}} \ for\ \alpha > 0.$$

Derive the coefficient of lower tail dependence for the Claydon copula. [**10**]

Solution:
Coefficient of lower tail dependence in terms of the copula function is

$$\lambda_L = \lim_{u \to 0^+} \frac{C[u, u]}{u}.$$

Setting $u = v$, we have

$$C[u, u] = (2u^{-\alpha} - 1)^{-\frac{1}{\alpha}}$$

so

$$\begin{aligned}\lambda_L &= \lim_{u \to 0^+} \frac{(2u^{-\alpha} - 1)^{-\frac{1}{\alpha}}}{u} \\ &= \lim_{u \to 0^+} (2 - \frac{1}{u^{-\alpha}})^{-\frac{1}{\alpha}} \\ &= 2^{-\frac{1}{\alpha}}\end{aligned}$$

(c) Let X and Y be two random variables representing the future lifetimes of two 30-year old individuals, who are married and live together. You are given that:

$$P(X \leq 40) = 0.16, P(Y \leq 40) = 0.25.$$

Calculate the joint probability that both lives will die by the age of 70 using the Clayton copula with $\alpha = 0.5$. [**5**]

Solution:

$$\begin{aligned}P(X \leq 40, Y \leq 40) &= C(u, v), where\ u = P(X \leq 40), v = P(Y \leq 40) \\ &= (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha} \\ &= (0.16^{-1/2} + 0.25^{-1/2} - 1)^{-2} \\ &= 0.0816\end{aligned}$$

(d) Is it appropriate if we use the Clayton copula with a new α value, $\alpha = -0.3$, to calculate the joint probability that the two lives in (c) will die by a certain age, with all else being the same? [**3**]

Solution:

The Clayton copula describes an interdependence structure in which there is lower tail dependence (the level of which is determined by the parameter α), but there is no upper tail dependence.
If $\alpha > 0$, there is lower tail dependence. If $-1 \le \alpha < 0$, there is no lower tail dependence.
Because the two lives are married and live together, their lives are dependent. it is not appropriate to have $\alpha = -0.3 < 0$.

(e) Now consider the **Frank** copula. The **Frank** copula is an example of Archimedean copulas. For the case where there are 3 variables, Archimedean copulas take the form:

$$C[u, v, w] = \Psi^{[-1]}(\Psi(u) + \Psi(v) + \Psi(w))$$

Derive an expression for the **Frank** copula for the case where the parameter $\alpha \neq 0$ and there are 3 variables.
The Frank copula has a generator function:

$$\Psi(t) = -ln\left(\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}\right)$$

[**10**]

Solution:

We need to find $\Psi^{[-1]}$, the pseudo-inverse generator function. Check

$$\Psi(0) = \lim_{t \to 0} -ln\left(\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}\right) = \infty$$

So, the pseudo-inverse $\Psi^{[-1]}$ is equal to the 'ordinary' inverse $\Psi^{(-1)}$ [**3**].

Let $y = \Psi^{(-1)}(x)$, then $\Psi(y) = x$. So

$$-ln\left(\frac{e^{-\alpha y} - 1}{e^{-\alpha} - 1}\right) = x$$

$$y = -\frac{1}{\alpha} ln\left[1 + (e^{-\alpha} - 1)e^{-x}\right]$$

$$\Psi^{[-1]}(x) = -\frac{1}{\alpha} ln\left[1 + (e^{-\alpha} - 1)e^{-x}\right]$$

Therefore

$$C[u, v, w] = \Psi^{[-1]}(\Psi(u) + \Psi(v) + \Psi(w))$$

$$= -\frac{1}{\alpha} ln\left[1 + (e^{-\alpha} - 1)e^{-[-ln(\frac{e^{-\alpha u}-1}{e^{-\alpha}-1}) - ln(\frac{e^{-\alpha v}-1}{e^{-\alpha}-1}) - ln(\frac{e^{-\alpha w}-1}{e^{-\alpha}-1})]}\right]$$

$$= -\frac{1}{\alpha} ln\left[1 + (e^{-\alpha} - 1)^{-2}(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)(e^{-\alpha w} - 1)\right]$$

Part a from lecture, part b, c similar to seminar, part d, e application of lecture material.
IFoA CS2 syllabus areas 1.3.1, 1.3.3.

Question 4 [25 marks]. Assume individual claim amounts, X, follow a distribution with density function

$$f(x) = 0.02xe^{-x}, \quad x > 0.$$

Claim events on a portfolio of insurance policies follow a Poisson process with parameter λ.
The insurance company calculates premiums using a premium loading of 30%.

(a) Derive the moment generating function, $M_X(t)$, given that $t < 1$. **[10]**

(b) The adjustment coefficient, R, is the unique positive root of the following equation, where $c = (1+\theta)\lambda E(X)$ is the rate of premium income and λ is the Poisson parameter:

$$M_X(R) = 1 + \frac{cR}{\lambda}.$$

Determine the adjustment coefficient R. (Showing the equation in the simplest form, with the only unknown parameter being R, is enough to get full marks. You don't have to solve for all the Rs.) **[10]**

(c) Suppose instead that individual claims are now for a fixed amount of $E(X)$. Will the probability of ultimate ruin for this business increase, decrease or remain unchanged? Explain your answer. **[5]**

Solution

(a)

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx$$
$$= \int_0^\infty 0.02xe^{(t-1)x}dx$$

We use integration by parts to solve the above. Let

$$u = 0.02x$$
$$\frac{dv}{dx} = e^{(t-1)x}$$

Then

$$\frac{du}{dx} = 0.02$$
$$v = \int e^{(t-1)x}dx = \frac{e^{(t-1)x}}{(t-1)}$$

So

$$M_X(t) = [uv]_0^\infty - \int_0^\infty v\frac{du}{dx}dx$$

$$= \left[\frac{0.02xe^{(t-1)x}}{t-1}\right]_0^\infty - \int_0^\infty \frac{0.02e^{(t-1)x}}{t-1}dx$$

$$= 0 - 0 - \left[\frac{0.02e^{(t-1)x}}{(t-1)^2}\right]_0^\infty$$

(provided that $t < 1$)

$$= \frac{0.02}{(t-1)^2}$$

(b)

$$M_X(R) = 1 + \frac{cR}{\lambda}$$

$$= 1 + (1+\theta)E(X)R$$

$$= 1 + (1.3)E(X)R$$

But:

$$E(X) = M'_X(0) = \frac{d}{dt}\left[\frac{0.02}{(t-1)^2}\right]_{t=0}$$

$$= \left.\frac{-2 \times 0.02}{(t-0.01)^3}\right|_{t=0} = 0.04$$

So we need to solve

$$\frac{0.02}{(R-1)^2} = 1 + 1.3 \times 0.04R$$

The following calculation of R is not required to get full marks.

$$R = -19.2298, R = 0.8616, R = 1.1374.$$

Since $M_X(t)$ in this question is valid for $t < 1$, we take the positive solution which is < 1, i.e.

$$R = 0.8616$$

(c) Probability of ultimate ruin will decrease.

In both scenarios, average claim amounts are the same.

However, the claim amounts in the first scenario are more variable / more uncertain, suggesting a greater risk.

or

In the first scenario, X can take values from $(0, \infty)$ and so have a higher variance compared to the second scenario where X can only take the value of 0.04.

Part a, c application of lecture material, part b similar to seminar.
IFoA CM2 syllabus areas 5.1.5.

End of Paper.