MTH6140 Linear Algebra II

Coursework 9

1. (a) What is the characteristic polynomial of the matrix A of Question 3 of the previous sheet? The matrix is repeated here for convenience:

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 2 & 3 & 1 \\ 4 & 2 & 4 \end{bmatrix}.$$

By trial and error (as described at the end of Section 5.5 of the notes) determine the minimal polynomial of A. Verify that the minimal polynomial is indeed a product of distinct linear factors, as asserted by Theorem 5.20.

(b) Repeat part (a) but with the matrix

$$B = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

That is, find the characteristic and minimal polynomials of B. Is B diagonalisable?

2. Suppose the linear map α on $(\mathbb{F}_2)^n$ is diagonalisable. What are the possible minimal polynomials of α? (There are not very many!) Deduce that α is a projection.

3. Suppose that linear map α over $\mathbb{C}^n$ is represented by $n \times n$ matrix A in Jordan form. Describe the matrix representing α^2. (Try this with a 3×3 block first, then generalise.)

4. Suppose A and B are real $n \times n$ matrices. Which of the following are true in general and which false?

(a) $\mathrm{Tr}(AB) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$.

(b) $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$.

(c) $\mathrm{Tr}(A^{-1}) = \mathrm{Tr}(A)^{-1}$.

In each case, either justify the claim or provide a counterexample.

5. Suppose that some linear map on $\mathbb{R}^3$ is represented by the matrix

$$A = \begin{bmatrix} 0 & 5 & -3 \\ 1 & -2 & 1 \\ 1 & -5 & 4 \end{bmatrix}.$$

(a) Compute the determinant $\det(A)$ and the trace $\mathrm{Tr}(A)$ of A.

(b) Compute the characteristic polynomial $p_A(x)$ of A and verify that that the coefficient of x^2 in $p_A(x)$ is $-\mathrm{Tr}(A)$, and that the constant coefficient in $p_A(x)$ is $(-1)^3 \det(A)$. (Compare this finding with Proposition 5.28.)

(c) Recall that the eigenvalues of A are the roots of $p_A(x)$. Verify that the product of the eigenvalues of A is $\det(A)$ and that the sum of the eigenvalues is $\mathrm{Tr}(A)$. (Again, compare this finding with Proposition 5.28.)

6. *Harder.* Suppose that π is a projection on a vector space V over $\mathbb{R}$ (or indeed over any field of characteristic 0). Prove that $\mathrm{Tr}(\pi) = \dim(\mathrm{Im}(\pi))$.

7. Let V_n be the vector space of real polynomials of degree at most $n - 1$. If $p(x)$ and $q(x)$ are polynomials in V_n, define $p \cdot q$ by

$$p \cdot q = \int_0^1 p(x)q(x)\,dx.$$

Check that "$\cdot$" is an inner product on V_n.