MTH6140 Linear Algebra II

Coursework 8

1. Let V_n and D be as in Question 6 of Assignment 6. So V_n is the vector space of real polynomials of degree at most $n-1$, and D is the linear map on V_n mapping each polynomial to its derivative.

(a) Let $n = 4$. What are the eigenvalues and associated eigenvectors of D as a map from V_4 to itself? (So, we are asking: what are the possible polynomials f, of degree at most 3, such that $f'(x) = \lambda f(x)$ for some $\lambda \in \mathbb{R}$?) As in Assignment 6, write down a general polynomial of degree at most 3 and see what constraints the coefficients must satisfy.

(b) Now let $\widehat{D}$ be the map on the vector space of real polynomials that maps each polynomial $f(x)$ to $xf'(x)$. What are the eigenvalues and associated eigenvectors of $\widehat{D}$ as a map from V_4 to itself? (Again, we are asking: what are the possible polynomials f, of degree at most 3, such that $xf'(x) = \lambda f(x)$ for some $\lambda \in \mathbb{R}$?)

(c) Which of the linear maps D and $\widehat{D}$ are diagonalisable? In cases where the map is diagonalisable, write down a basis composed of eigenvectors, and the matrix representing the map relative to that basis.

2. The linear map $\alpha : V \to V$ satisfies (i) α is a projection, and (ii) α is invertible. Determine α.

3. If you successfully completed Question 6 of the previous assignment, you will have calculated eigenvectors of the following real matrix:

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 2 & 3 & 1 \\ 4 & 2 & 4 \end{bmatrix}.$$

If not, a possible solution is: eigenvectors $v_1 = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^\top$ and $v_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^\top$ with eigenvalue $\lambda_1 = 2$, and $v_3 = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^\top$ with eigenvalue $\lambda_2 = 3$.

(a) Find matrices P and $Q = P^{-1}$ such that QAP is the diagonal matrix

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(Recall that the columns of P can be taken to be eigenvectors of A, in the correct order.)

(b) Note that $A = PDP^{-1}$. What is the limit of the sequence of matrices $3^{-t}A^t$ as $t \to \infty$?

4. True or false?

(a) Let A be an $n \times n$ matrix with real entries. If A is diagonalisable viewed as a linear map on $\mathbb{C}^n$ then it is diagonalisable viewed as a linear map on $\mathbb{R}^n$.

(b) If A is diagonalisable viewed as a linear map on $\mathbb{R}^n$ then it is diagonalisable viewed as a linear map on $\mathbb{C}^n$.

(c) If A has n distinct real eigenvalues then it is diagonalisable viewed as a linear map on $\mathbb{R}^n$.

In each case, either justify the claim or provide a counterexample. Theorem 5.20 may be useful.

5. Suppse α is a linear map on $\mathbb{R}^2$. For each of (a)–(c), write down a 2×2 matrix A such that the linear map α represented by A has minimal polynomial $m_\alpha(x)$, where

(a) $m_\alpha(x) = x - 1$,

(b) $m_\alpha(x) = (x-1)^2$,

(c) $m_\alpha(x) = x^2 - 2x + 2$.