L7 Announcements ① Learning Support hour → Learning Café

Wed 2-3pm

MB-B11

② Undergrad Research Seminar

Wed 3pm MB-204

[$\subsetneq$ means $\subseteq$ but not $=$]

Finish May 2019 Exam Q1 $U, W \subseteq V$ subspaces

(e) Show $\dim(U+W) \geq \dim(U), \dim(W)$

Ans: $U+W \supseteq U$, $\supseteq W$ $u \in U$ can written as $\underset{U}{u} = u + \underset{W}{0} \in U+W$

$w \in W$ " " " $w = \underset{U}{0} + \underset{W}{w} \in U+W$

$\Rightarrow \dim(U+W) \geq \dim(U), \dim(W)$

(since a basis of the smaller subspace is l.i. viewed in the bigger one $\therefore$ can be extended to a basis of the bigger one)

(f) V has $\dim(V) = n > 2$, $U, W \subseteq V$ have $\dim(U) = \dim(W) = n-1$. What are $\dim(U+W)$, $\dim(U \cap W)$? Told $U \neq W$

$\exists v \in W \setminus U$ (or $w \in U \setminus W$)

let $u_1 \dots u_{n-1}$ be a basis of U. $v \notin U$ so

$u_1 \dots u_{n-1}, v$ are l.i. (since if a linear relation $cv + \sum c_i u_i = 0$ $\Rightarrow v \in \langle u_i \rangle = U$ so not possible)

Also $\langle u_1, \dots u_{n-1}, v \rangle \subseteq U+W$

$\therefore$ ~~dim~~ $\dim(U+W) \geq \dim(\langle u_1, \dots u_{n-1}, v \rangle) = n$

but $U+W \subseteq V$ so $\dim(U+W) \leq \dim(V) = n$ $\therefore \dim(U+W) = n$

$\therefore \dim(U \cap W) = \dim(U) + \dim(W) - \dim(U+W) = n-1+n-1-n = n-2$

We saw last time that if $U, W \leq V$ are subspaces then every element of $U \oplus W$ has a unique description as $u + w$, $u \in U$, $w \in W$

<u>Definition 1.26</u> Let $U_1, \dots, U_r \leq V$ be subspaces. We say V is a direct sum and write

$$V = U_1 \oplus U_2 \oplus \dots \oplus U_r$$

if every $v \in V$ can be written uniquely as

$$v = u_1 + u_2 + \dots + u_r, \quad u_i \in U_i \quad \forall i = 1, \dots, r$$

[check that this is equivalent to $U_1 \oplus U_2$ in case $r = 2$ as defined previously as $U_1 \cap U_2 = \{0\}$]

<u>Lemma 1.27</u> Suppose $U_1, \dots, U_r \leq V$ are subspaces and $V = U_1 + U_2 + \dots + U_r$. Then TFAE

(a) V is a direct sum of the U_i

(b) If $u_1 + \dots + u_r = \underline{0}$ with $u_i \in U_i$, $i = 1, \dots, r$ then $u_1 = u_2 = \dots = u_r = \underline{0}$

<u>Proof</u> (a) $\Rightarrow$ (b) Let $u_1 + \dots + u_r = \underline{0}$. Then

$$\underline{0} \in V \text{ so } \underline{0} = \underset{\substack{\cap \\ U_1}}{\underline{0}} + \underset{\substack{\cap \\ U_2}}{\underline{0}} + \dots + \underset{\substack{\cap \\ U_r}}{\underline{0}}$$

$\therefore$ by uniqueness, $u_i = 0$.

(b) $\Rightarrow$ (a): Suppose $v \in V$ and

$$v = u_1 + \dots + u_r = u_1' + \dots + u_r'$$

$$\Rightarrow \underset{\substack{\cap \\ U_1}}{(u_1 - u_1')} + \dots + \underbrace{(u_r - u_r')}_{\in U_r} = 0$$

$\therefore$ by assumption (b) $\Rightarrow u_1 - u_1' = 0, \dots, u_r - u_r' = 0$

$\therefore$ (a) holds

QED

Note the similarity with the notion of l.i. In fact [Q1 of CWK3] $v_1 \dots v_n$ is a basis of V iff $V = \langle v_1 \rangle \oplus \dots \oplus \langle v_n \rangle$

<u>Lemma 1.28</u> If $V = U_1 \oplus \dots \oplus U_r$ then

(a) If B_i is a basis of U_i, $i = 1 \dots r$ then the combined list $B_1, B_2 \dots B_r$ is a basis of V

(b) $\dim(V) = \dim(U_1) + \dots + \dim(U_r)$

<u>Proof</u> Clearly (a) $\Rightarrow$ (b). For (a), we already know that $V = U_1 + \dots U_r$ so every element $v \in V$ can be written as $v = u_1 + \dots + u_r$. Each u_i can be expanded in basis B_i so

$B = B_1, \dots B_r$ clearly spans.

Now suppose a linear combination of B is zero

let $d_i = \dim(U_i)$, $B_i = u_{i,1}, \dots, u_{i,d_i}$ so

if $u_1 + \dots + u_r = 0$, $u_i = a_{i,1} u_{i,1} + \dots a_{i,d_i} u_{i,d_i}$

for some coefficients $a_{i,j}$. By lemma 1.27

$u_1 = u_2 \dots = u_r = 0$ as $V = U_1 \oplus \dots \oplus U_r$

Each $u_{i,1} \dots u_{i,d_i}$ are a basis so

$a_{i,1} = \dots = a_{i,d_i} = 0$ for $i = 1 \dots r$

$\therefore$ B is l.i. hence a basis QED

Quiz / challenge If $V = U_1 \oplus \cdots \oplus U_r$

is it true that $U_i \cap U_j = \{0\}$ $\forall i \neq j$

☑ yes ☐ no ☐ depends

If $u \in U_i \cap U_j$ $\quad \underset{U_i}{u} + \underset{U_j}{(-u)} = 0$ $\quad \therefore$ by Lemma 1.27 $u = 0$

Challenge if $V = U_1 + \cdots + U_r$ and

$U_i \cap U_j = \{0\}$ $\forall i \neq j$ $\quad r > 2$

is $V = U_1 \oplus \cdots \oplus U_r$?

Reminder Quiz deadline 11.59pm thursday.

L8

Chapter 2 Matrices

2.1 (Revision) We'll work with $m \times n$ matrices

$\begin{array}{cc} m & n \\ \uparrow & \uparrow \\ \#\text{rows} & \#\text{cols} \end{array}$

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$ entries in $\mathbb{K}$

(eg square matrices $m = n$, set of these is $M_n(\mathbb{K})$)

The space of $m \times n$ matrices is v.s.

If $A = (a_{ij})$, $B = (b_{ij})$ then

$A + B = C = (c_{ij})$: $c_{ij} = a_{ij} + b_{ij}$

and cA has entries $cA = (ca_{ij})$ for $c \in \mathbb{K}$. Also, if A is $m \times n$ and B is $n \times p$ then

$$AB = C = (c_{ij}), \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$i = 1 \dots m, \; j = 1 \dots p$$

is an $m \times p$ matrix.

(So $M_n(\mathbb{K})$ is an algebra with $AB \neq BA$ typically)

Special matrices $\underline{0} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

and if $m = n$, $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $n \times n$ identity matrix

2.2 Row and Col ops

__Definition__ Elementary row ops on an $m \times n$ matrix are:

1. add a multiple of jth row to ith row $(j \neq i)$
2. Scale the i'th row by $a \in \mathbb{K}$, $a \neq 0$
3. Swap ith and jth rows $(i \neq j)$

Elementary column ops defined similarly replacing "row" by "column".

Associated to every row or col op is an "elementary matrix" defined by applying the op to identity matrix I_m for a row op, I_n for col op.

<u>Example</u> $m=3$, row op "add twice the 2nd row to the 1st row of a $3\times n$ matrix" (r_1+2r_2 shorthand notation)

The associated elementary matrix is

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{r_1+2r_2} \begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}$$

This is also the elementary matrix for the col op on an $m\times 3$ matrix "add twice the first col to the 2nd col" (c_2+2c_1)

<u>Lemma 2.5</u> The effect of a row op is given by multiplying the $m\times n$ matrix on the <u>left</u> by the associated elementary matrix. The effect of a col op is given by multiplying on the <u>right</u> by the associated elementary matrix.

<u>Example</u> $A=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$ 2×3 matrix

row op: subtract 4 times 1'st row from 2nd

$r_2-4r_1 \Rightarrow R=\begin{bmatrix}1&0\\-4&1\end{bmatrix}$ elementary matrix.

col op: subtract 2 times col 1 from col 2

$c_2-2c_1 \Rightarrow$

$$C = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ elementary matrix.

Applying to left and right,

$$RAC = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -3 & -6 \end{bmatrix}$$

check $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{r_2 - 4r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow{c_2 - 2c_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -3 & -6 \end{bmatrix}$ " ✓

1) Also note $(RA)C = R(AC)$ i.e. you can interchange the order of row with col ops.

2) Note elementary matrices are all invertible since row/col ops are reversible. E.g. $c_2 + 2c_1$ has matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ can be reversed by $c_2 - 2c_1$

so $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ as elementary matrices ✓

2.3 Rank of an $m \times n$ matrix

Def 2.7 Let A be an $m \times n$ matrix over $\mathbb{K}$

Define:

Row space $(A) :=$ subspace of $\mathbb{K}^n$ spanned by the rows.

Col space $(A) :=$ " " $\mathbb{K}^m$ " " " cols.

rowrank $(A) := \dim(\text{Row space}(A))$

Col rank $(A) := \dim(\text{Col space}(A))$

here row rank (A) = max # of l.i. rows
col rank (A) = max # of l.i. columns

Lemma 2.9 (a) Elementary col ops preserve colspace (A)
(and hence ~~col~~ rank (A))

(b) Elementary row ops preserve Rowspace (A)
(and hence row rank (A))

(c) Elementary row ops preserve colrank (A)

(d) " col " " rowrank (A)

[Note (c), (d) do not say that these preserve the colspace (A) and ~~of~~ row space (A)]

proof Let v_i be the columns of $A = \begin{bmatrix} \begin{bmatrix} \vdots \\ v_1 \\ \vdots \end{bmatrix} \cdots \begin{bmatrix} \vdots \\ v_n \\ \vdots \end{bmatrix} \end{bmatrix}$

if $u = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ then $Au = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \sum_i c_i v_i$

so a linear relation
$\sum c_i v_i = 0 \iff u$ s.t. $Au = \underline{0}$

and colspace $(A) = \langle v_1, \dots v_n \rangle$
$= \{ \sum_i c_i v_i \} = \{ Au \mid u \in \mathbb{K}^n \}$

(a) If $v \in$ colspace $(A) \Rightarrow v = Au$ some $u \in \mathbb{K}^n$

Consider a col op with matrix C so a new matrix $A' = AC$. Then
$v = AC(C^{-1}u) = A'u'$ some other $u' = C^{-1}u$

$\Rightarrow \quad v \in$ col space (A')

So col space $(A) \subseteq$ col space (A')

But C is invertible so can consider

$A = A'C^{-1}$ as obtained from A'

$\therefore$ by same result with A, A' swapped,

col space $(A') \subseteq$ col space (A)

so col space (A') = col space (A).

(b) Analogous to (a) w. in rows in place of columns $A' = RA$ some elementary row op matrix R etc.

(c) let $A' = RA$ for some elementary matrix R.

col space (A') is spanned by the cols of A'

$\therefore$ by theorem 1.15 (b) $\exists$ a subset of rows $v_1' \dots v_k'$ forming a basis,

where $k =$ dim (col space (A')) = colrank(A')

let $v_i = R^{-1} v_i'$ $i = 1 \dots k$ $\in$ col space (A)

These are l.i. since $\sum_{i=1}^{k} c_i v_i = 0$

$\Rightarrow \sum_{i=1}^{k} c_i \underbrace{R v_i}_{v_i'} = 0 \quad \Rightarrow \quad c_i = 0$ as v_i' basis.

So colrank $(A) \geq k$ = colrank (A').

Now regard A' as obtained from A by R^{-1}

$\Rightarrow$ colrank $(A') \geq$ colrank (A) (swapping roles A, A')

$\therefore$ colran (A') = colrank (A)

(d) analogous to (c) by swapping roles of ~~cols~~ rows and colums. Q.E.D.

Theorem 2.10 Any $m \times n$ matrix A over $\mathbb{K}$ can be transformed by row and col ops to a matrix (the "canonical form for equivalence") of the form

$$D = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

for some $r \geq 0$.

$\boxed{L9}$ proof Let $A = (a_{ij})$. As induction hypothesis, we assume the result for all matrices smaller than $m \times n$. If $A = 0$ we are done, $r = 0$. Hence assume $A \neq 0$. If $a_{11} = 0$ find some $a_{ij} \neq 0$ and swap 1st row with i'th and 1st col with j'th to bring this to the $(1,1)$ position. Hence assume w.l.o.g $a_{11} \neq 0$. We can also scale the first row by a_{11}^{-1} (a row op) to make this entry 1. So wlog assume $a_{11} = 1$.

Now use type 1 moves to set all the other entries in the 1st row and 1st col to 0.

(If $a_{1j} \neq 0$ subtract $a_{1j} \times$ 1st col from the jth i.e. $c_j - a_{1j} c_1$

$$\begin{bmatrix} 1 & \cdots & a_{1j}^{\neq 0} & \cdots \\ \vdots & & & \\ \vdots & & & \end{bmatrix} \to \begin{bmatrix} 1 & \cdots & 0 & \cdots \\ \vdots & & \vdots & \\ \vdots & & & \end{bmatrix}$$

etc)

We end up with a matrix of the form

$$\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right]$$

for some smaller matrix A'.

Assume the result for A'

$\Rightarrow$ the result for A after using row/col ops (which dont affect the 1st row or 1st col) to put A' in canonical form, so

$$A = \left[\begin{array}{c|cc} 1 & 0 \cdots & 0 \\ \hline 0 & \begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix} & 0 \\ \vdots & & \\ 0 & 0 & 0 \end{array}\right] \quad (\text{or } A' = 0) \quad .\quad \text{Q.E.D.}$$

<u>Corollary 2.12</u> For any $m \times n$ matrix A

the rowrank(A) = colrank(A) =: "the rank" rank(A)

<u>Proof</u> We put A into canonical form for equivalence

$$D = \left[\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right] = \left[\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{matrix}}^{r} \Big\} r & O \\ \hline O & O \end{array}\right]$$

which has colrank$(D) = r$ = rowrank(D)

But colrank(A) = colrank(D) = rowrank(D) $(= r)$
= rowrank(A)

Since we saw that row and col ops dont change either row or col ranks (lemma 2.9).

Q.E.D.

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ as matrix over $\mathbb{R}$.

We already have $a_{11} = 1$ so skip the first part of the proof/procedure to find D via the theorem.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{r_2 - 4r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow{c_2 - 2c_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & -3 & -6 \end{bmatrix}$$

$$\xrightarrow{c_3 - 3c_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{-3 \; -6} \end{bmatrix} \xrightarrow{-\frac{1}{3} r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1 \; 2} \end{bmatrix} \xrightarrow{c_3 - 2c_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = D$$

A'

$\therefore$ $\operatorname{rank}_{\mathbb{R}}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = 2$ $(r = 2)$

either row rank or col rank.

Quiz Can the rank of a matrix depend on the field?

☑ yes ☐ no ☑ depends

Example $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ over $\mathbb{F}_3$ is $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$

$$\xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{c_2 - 2c_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\operatorname{rank}(A) = 1$ over $\mathbb{F}_3$ (over $\mathbb{R}$ it was 2)

Theorem 2.14 For any $m \times n$ matrix A over $\mathbb{K}$ $\exists$ invertible matrices P $(m \times m)$ and Q $(n \times n)$ s.t. $D = PAQ$ is the canonical form for equivalence. Moreover P, Q are products of elementary row, col op matrices, respectively.

Proof Write $R_1, R_2 \dots R_s$ for the elementary row op matrices and $C_1, \dots C_t$ for the elementary col op matrices for the operations in Theorem 2.10. So

$$D = \underbrace{R_s \cdots R_2 R_1}_{P} A \underbrace{C_1 C_2 \cdots C_t}_{Q}$$

Q.E.D.

In our above example $R_1 = r_2 - 4r_1$ applied to I_2 $= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

$C_1 = c_2 - 2c_1$ applied to I_3 $= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $R_2 = -\frac{1}{3} r_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$

$C_2 = c_3 - 3c_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C_3 = c_3 - 2c_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

Check $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = R_2 R_1 A C_1 C_2 C_3$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}_{} A \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{}$$

$P = \begin{bmatrix} 1 & 0 \\ 4/3 & -\frac{1}{3} \end{bmatrix}$ $\qquad Q = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(Check you get $D = PAQ$).

A quick way to get P, Q is to apply the same row / col ops to the relevant identity matrices. eg

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{c_2 - 2c_1} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{c_3 - 3c_1} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{c_3 - 2c_2} \underbrace{\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{Q}$$

similarly apply row ops to I_2 to get P.

Def 2.16 $m \times n$ matrices A, B are said to be *equivalent* if $B = PAQ$ for some invertible $P \in M_m(\mathbb{k})$, $Q \in M_n(\mathbb{k})$

Check this is an equivalence relation – i.e. reflexive, symmetric, transitive.

So theorem 2.14 says that every A is equivalent to a matrix of form D (hence the name)

Announcements
① Quiz due 11.59 thursday
[Advise not to "wing it" but do the questions in advance]
② Learning café B11 (maths building) 2pm
③ Undergrad research seminar (3pm?).
④ Next monday tutorial $\rightarrow$ Quiz 1 answers.