Quiz 2

Q 1-6) $f_{X,Y}(x,y) = \begin{cases} 1.4x + 0.6y & 0<x<1,\ 0<y<1 \\ 0 & \text{else} \end{cases}$

$$f_X(x) = 1.4x + 0.6\int_0^1 y\,dy = 1.4x + 0.6\,\frac{y^2}{2}\Big|_0^1 = 1.4x + 0.3$$

$\Rightarrow A = 1.40 \qquad B = 0.3$

$$f_{Y|X=\frac{1}{3}}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}\Big|_{x=\frac{1}{3}} = \frac{1.4\cdot\frac{1}{3} + 0.6y}{1.4\cdot\frac{1}{3}+0.3} = \underbrace{0.608}_{D} + \underbrace{0.783}_{G}\,y$$

$$E(X) = \int_0^1 x(1.4x+0.3)\,dx = 1.4\,\frac{x^3}{3}\Big|_0^1 + 0.3\,\frac{x^2}{2}\Big|_0^1 = 0.617$$

X, Y not independent (see definition of independence in lectures)

Q 7-8) $Var(X) = 0.7 \quad Cov(X,Y) = 0.4 \quad Cov(X,Z) = 1.2 \quad Cov(Y,Z) = 0.8$

$$Cov(12Y, 9X) = 12\cdot 9\cdot Cov(X,Y) = 43.2$$

$$\begin{aligned} Cov(12Y+3, 9X+8Z) &= Cov(12Y, 9X+8Z) \\ &= Cov(12Y, 9X) + Cov(12Y, 8Z) \\ &= 43.2 + 12\cdot 8\cdot Cov(Y,Z) = 120 \end{aligned}$$

Q 9-11)

$W = aX + (1-a)Y \qquad a = 0.8 \qquad E(X) = 1 \quad E(Y) = 3$

$Var(X) = 3 \quad Var(Y) = 7$

$$\begin{aligned} E(12W) &= 12\,E(W) \\ &= 12\big(a\,E(X) + (1-a)\,E(Y)\big) = 34.08 \end{aligned}$$

$$Var(W) \underset{\text{independence}}{\overset{\uparrow}{=}} Var(aX) + Var((1-a)Y) = a^2 Var(X) + (1-a)^2 Var(Y) = 5.944$$

$$\Rightarrow Var(12W) = 12^2\cdot 5.944 = 855.94$$

Assume $Cov(X,Y) = -1.7$

$$\Rightarrow Cov(aX, (1-a)Y) = a(1-a)\,Cov(X,Y) = 0.125$$

$$\Rightarrow Var(W) = 5.944 + 2a(1-a)\underbrace{Cov(X,Y)}_{-1.7} = 5.694$$

$$\Rightarrow Var(12W) = 12^2\cdot 5.694 = 819.9$$

Q 12-15) $f_X(x,y) = \begin{cases} 2-x-y & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{else} \end{cases}$

$$\begin{aligned} E(X) &= \int_0^1\!\int_0^1 (2-x-y)\,x\,dx\,dy = 2\,\frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1\,\frac{y^2}{2}\Big|_0^1 \\ &= 1 - \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{5}{12} = 0.417 \end{aligned}$$

$$E(X^2) = \int_0^1\int_0^1 (2-x-y)\, x^2\, dx\, dy = \int_0^1 2x^2 dx - \int_0^1 x^3 dx - \int_0^1 x^2 dx \int_0^1 y\, dy$$

$$= 2\frac{x^3}{3}\Big|_0^1 - \frac{x^4}{4}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 \; \frac{y^2}{2}\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{4} - \frac{1}{3}\cdot\frac{1}{2} = \frac{8-3-2}{12} = \frac{3}{12} = \frac{1}{4}$$

$$\Rightarrow \operatorname{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144} = 0.0764$$

Similarly $\operatorname{Cov}(X,Y) = E(XY) - E(X)\,E(Y) = \dots \approx -0.007$

$$\operatorname{corr}(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} = \dots = -0.091$$

Q17 – 20)

$$f_{X,Y}(x,y) = \begin{cases} 300.125\cdot(x^2-y^2)\, e^{-7x} & 0<x<\infty,\ -x<y<x \\ 0 & \text{else} \end{cases}$$

$$f_X(x) = \int_{-x}^{x} 300.125\cdot(x^2-y^2)e^{-7x} dy = 300.125\cdot x^2 e^{-7x}\cdot 2x - \frac{y^3}{3}\Big|_{-x}^{x}\cdot 300.125\, e^{-7x}$$

$$= 300.125\left(2x^3 e^{-7x} - \frac{2}{3}x^3 e^{-7x}\right)$$

$$= 300.125\cdot\frac{4}{3}x^3 e^{-7x} = 400.17\cdot x^3 e^{-7x}$$

$\Rightarrow A = 400.17 \qquad B = 3$

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \underbrace{\frac{300.125}{400.17}}_{0.75}\,\frac{(x^2-y^2)}{x^3}$$

$\Rightarrow \quad C = 0.75$

$D = -3$

$E = 0$