MTH5129 Probability & Statistics II

Coursework 1

1. In this question we revise some standard probability distributions (i.e. random variables). For each of the following distributions write down its *probability mass function*, its *mean*, its *variance* and a *description of the experiment* related to it.

   a) **Bernoulli distribution** with parameter p

   b) **Binomial distribution** with parameters n and p

   c) **Geometric distribution** with parameter p

   d) **Poisson distribution** with parameter λ

2. In this question, we move on to the revision of some "common" continuous random variables

   a) **Uniform Distribution.** A random variable X has Uniform distribution on $[a, b]$ and write $X \sim U(a, b)$ if

   $$f_X(x) = \begin{cases} \frac{1}{(b-a)} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

   Prove that this probability density function integrates to one.

   b) **Exponential Distribution.** A random variable X has exponential distribution with parameter $\lambda > 0$ and write $X \sim Exp(\lambda)$ if

   $$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   Prove that this probability density function integrates to one.

   c) **Gamma Distribution.** A random variable X has a Gamma distribution with *shape* parameter $\alpha > 0$ and *rate* parameter $\beta > 0$ and we write $X \sim Ga(\alpha, \beta)$ if

   $$f_X(x) = \begin{cases} \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

   where $\Gamma(\alpha)$ is the Gamma function, which is given by

   $$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

   Prove that this probability density function integrates to one.

Remark 1 *Note that in some textbooks, the Gamma distribution can be defined with* $\beta = 1/\theta$ *for the scale parameter* θ *instead, or write it as* $Ga(\beta, \alpha)$. *You always have to make sure what each parameter stands for and know what notation is used in each case.*

d) **Chi-Square.** A random variable X has a Chi-Square distribution with ν *degrees of freedom* and we write $X \sim \chi^2(\nu)$, if X has a $Ga(\nu/2, 1/2)$ distribution for some integer $\nu \in \mathbb{N}$. In such a case,

$$f_X(x) = \begin{cases} \frac{x^{\nu/2-1}\, e^{-x/2}}{2^{\nu/2}\,\Gamma(\nu/2)} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

Using the above probability density function, and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (not shown here), it is easy to see that, the probability density function of $X \sim \chi^2(1)$ is

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi x}}\, e^{-x/2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

Prove that this probability density function integrates to one.

e) **Normal Distribution.** A random variable X has Normal distribution with parameters (μ, σ), for $\sigma > 0$, and write $X \sim N(\mu, \sigma^2)$ if

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$

We shall use the fact that the Normal probability density function integrates to 1 (the proof is a bit technical – involves passing to polar coordinates – not shown in this course – non-examinable). It is however useful to observe that by a simple change of variable in the integral, namely $y = (x - \mu)/\sigma$ we obtain

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy$$

We now see that the integral in the left hand side of the above formula does not depend on the parameters (μ, σ)!

f) **(Student's) t Distribution.** A random variable X has t distribution with ν degrees of freedom and we write $X \sim t_\nu$ if

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

We shall use the fact that the above probability density function integrates to 1 (non-examinable).

Remark 2 *Some books call this distribution Student's t, because the first person to derive it published his result under the pseudonym of "Student".*

g) **Cauchy Distribution.** A random variable X has Cauchy distribution with *location* parameter x_0 (specifying the location of the peak of the distribution) and *scale* parameter γ and we write $X \sim Cauchy(x_0, \gamma)$ if

$$f_X(x) = \frac{1}{\pi\gamma}\left[\frac{\gamma^2}{(x-x_0)^2+\gamma^2}\right],$$

It is an unusual distribution because it has no defined mean or variance (we will see why in the lecture notes).
Moreover if X is Cauchy then $1/X$ is also Cauchy (we will learn several methods for proving such type of statements).

3. The Gamma function Γ involved in $Ga(\alpha, \beta)$ distribution is given by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx.$$

(i). Prove that, for any $\alpha > 0$, we have

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1).$$

(ii). Prove that, for any integer $n \geq 1$, we have

$$\Gamma(n) = (n-1)!.$$

4. Given a normal random variable $X \sim N(\mu, \sigma^2)$, prove that $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

5. Suppose that there are two urns. The first contains 5 red balls, 3 green balls, and 2 blue balls. The second contains 2 red balls and 4 green balls.

   We pick a ball at random from the first urn and place it in the second urn. We then pick a ball at random from the second urn (which might be the ball we have just placed there).

   a) What is the probability this ball is red?
   b) What is the probability it is green?
   c) What is the probability it is blue?
   d) What is the expected number of trials of the above experiment until we finally pick a blue ball?

6. I roll two fair dice. Use the Theorem of Total Probability for Expectations to calculate the expected value of the product of the two numbers rolled.