MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra
(2023/2024)

COURSEWORK 9

WebWork submission of **exercise marked** $(*)$ due:
11.59am on Monday 20 December 2023

You should also attempt all of the other exercises in order develop your mathematical reasoning and skill in constructing arguments and proofs; model solutions will be posted on QMPlus after the submission date.

Exercise $(*)$ 1. Solve WeBWork **Set 9** at:

`https://webwork.qmul.ac.uk/webwork2/MTH5112-2023.`

Log in with your 'ah***' QMUL ID as username, and your student number as password, see Coursework 0 for further instructions.

Exercise 2. (a) Prove that for all vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, we have

$$||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2(\mathbf{x}\cdot\mathbf{y}). \tag{1}$$

and use this to deduce the Pythagorean Theorem in $\mathbb{R}^n$ (Proposition 7.8 from lectures), i.e. that vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are orthogonal if and only if

$$||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2.$$

(b) Use equation (1) to prove the *Cauchy–Schwartz inequality*, which says that

$$|\mathbf{u}\cdot\mathbf{v}| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$$

for all vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$. Hint: let $\mathbf{x} = ||\mathbf{u}||\mathbf{v}$ and $\mathbf{y} = -||\mathbf{v}||\mathbf{u}$ in (1), and note that both sides of (1) are non-negative.

(c) Use equation (1) and the Cauchy–Schwartz inequality to prove the *triangle inequality*, which says that

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

for all vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$. Hint: start by expanding $||\mathbf{u}+\mathbf{v}||^2$ using (1).

Exercise 3. Let H be a subspace of $\mathbb{R}^n$. Prove the following:

(a) $H^\perp$ is also a subspace of $\mathbb{R}^n$.

(b) If $H = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ then a vector $\mathbf{x} \in \mathbb{R}^n$ is an element of $H^\perp$ if and only if $\mathbf{x}$ is orthogonal to each of the spanning vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$.

(c) $\dim(H) + \dim(H^\perp) = n$. Hint: choose a basis for H and think of a way to use the rank–nullity theorem.

Exercise 4. Let H be the subspace of $\mathbb{R}^3$ spanned by the two vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

(a) Find a basis of $H^\perp$. (Hint: notice that $H^\perp$ is the nullspace of a certain 2×3 matrix.)

(b) Give geometric descriptions of H and $H^{\perp}$.

Exercise 5. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -4 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

(a) Show that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbf{R}^3$.

(b) Find the coordinate vectors of the the following vectors in the basis B:

$$\mathbf{u} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}.$$

Hint: do *not* solve any linear systems or compute the inverses of any matrices; instead, use the fact that B is an *orthogonal* basis and apply an appropriate theorem from Chapter 6 of the lecture notes.