MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 1

WebWork submission of **exercise marked** $(*)$ due:
11.59am on Wednesday 18 October 2023

You should also attempt all of the other exercises in order develop your mathematical reasoning and skill in constructing arguments and proofs; model solutions will be posted on QMPlus after the submission date.

Exercise $(*)$ 1. Solve WeBWork **Set 1** at:

`https://webwork.qmul.ac.uk/webwork2/MTH5112-2023/`.

Log in with your '`ah***`' QMUL ID as username, and your student number as password, see Coursework 0 for further instructions.

Exercise 2. Suppose that A and B are matrices of the same size. Prove the following:

(a) $(\alpha A)^T = \alpha A^T$ for all scalars α;
(b) $(A + B)^T = A^T + B^T$.

(Hint: you must give a general proof, not just consider specific examples. Use the definition of "transpose", namely that if the (i, j)-th entry of A is a_{ij}, then the (i, j)-th entry of A^T is a_{ji}.)

Exercise 3. (a) Suppose that A and B are square matrices of the same size. Prove that the condition

$$(A + B)^2 = A^2 + 2AB + B^2$$

is satisfied if and only if A and B commute. (Note that you are being asked for a general proof. It is not enough to just choose two specific matrices.)

(b) Suppose that A is a square matrix with the property that

$$A^3 = O$$

(here "O" means the zero matrix of the same size as A). Prove that the matrix $I + A$ is invertible, and that its inverse is given by

$$(I + A)^{-1} = I - A + A^2.$$

(Note that you cannot assume anything about A other than the condition $A^3 = 0$.)

(c) Suppose that A is an invertible matrix. Prove that A^T is also invertible, and that its inverse is given by

$$(A^T)^{-1} = (A^{-1})^T .$$

(Hint: you need to verify the condition in the definition of the "inverse" of a matrix. The property $(AB)^T = B^T A^T$ might be useful.)

Exercise 4. Let A and B be square matrices of the same size.

(a) Prove that if A and B are both diagonal, then AB is diagonal and A and B commute.
(b) Prove that if A and B are both upper triangular, then AB is upper triangular.
(c) If A and B are both upper triangular, do they necessarily commute? If you think they must commute, give a proof; if you think they might not commute, give a counterexample.

Exercise 5. Let A, B and C be square matrices of the same size.

(a) Prove that if A is symmetric then BAB^T is symmetric.
(b) Prove that if A and B are symmetric, then AB is symmetric if and only if A and B commute.
(c) Use the Invertible Matrix Theorem from lectures to prove that if $AB = I$, then also $BA = I$.
(d) Prove that if A is invertible and $AB = AC$, then $B = C$.

Exercise (MATLAB) 6. In this exercise you are asked to use the mathematical software MATLAB, which is available on college's computers. In order to use it on your personal computer, please follow the instructions at: `https://tinyurl.com/sheawvmv`.

Weekly coursework sheet may contain exercises intended to be completed using MATLAB. However, it is not my intention to tell you how to do everything. Often the best way to learn how to use a new piece of mathematical software is to actually use it; starting with basic things if necessary until you figure out everything you need to know. The exercises will give you some of the necessary code/commands needed to complete them, but you will also need to figure some things out for yourself by, e.g. consulting the help/documentation, finding similar examples online, figuring things out with your colleagues, and so on.

(a) Use the following MATLAB commands to define the matrices A and B from Exercise 2(a):

```
A = [2 4; -6 0]
B = [1 -5; -3 2]
```

Now figure out how to add and subtract matrices and how to multiply matrices by scalars, and compute $A - B$ and $\frac{1}{2}A - 3B$.

(b) Define a matrix by running the following command:

```
M = [1 1 1 -3 -2; 2 3 0 -4 1; -3 -4 -1 6 -1]
```

Now use the MATLAB function "`rref`" to put this matrix into reduced row echelon form; that is, run the following command:

```
rref(M)
```

(c) Use the function "`inv`" to compute the inverse of the matrix A. Give this matrix a name, e.g. call it C by running the command

```
C = inv(A)
```

Check directly that C is actually the inverse of A, i.e. check (using MATLAB) that multiplying the matrices together in either order gives the identity matrix.