EBU5375 Signals and systems:
Fourier series of discrete-time periodic signals

Dr Jesús Requena Carrión
Agenda

Quick review

The notion of frequency in discrete-time signals

Fourier series representation of discrete-time periodic signals

Important properties of Fourier series
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The notion of frequency in discrete-time signals

Fourier series representation of discrete-time periodic signals

Important properties of Fourier series
The notion of frequency in CT

If $x(t)$ has high-frequency components

(a) The amplitude of $x(t)$ is high.
(b) The amplitude of $x(t)$ changes quickly.
(c) Signal $x(t)$ contain many different frequency components.
LTI filters and the convolution theorem

\[ x(t), X(\omega) \xrightarrow{H(\omega)} y(t), Y(\omega) \]

\[ y(t) = x(t) \ast h(t) \xrightarrow{FT} Y(\omega) = X(\omega)H(\omega) \]

If \( H(\omega) \) is a high-pass filter

(a) Signal \( y(t) \) has the high-frequency components of \( x(t) \).

(b) Signal \( y(t) \) has new high-frequencies components that \( x(t) \) doesn't have.

(c) Both (a) and (b).
Nonlinear filters and the modulation theorem

\[ x(t) \xrightarrow{\times} y(t) = x(t)c(t) \]
\[ c(t) = \cos(\omega_0 t) \]

The frequency components of \( y(t) \)

(a) Have a **higher frequency** than the components of \( c(t) \).
(b) Have a **lower frequency** than the components of \( c(t) \).
(c) Can be **both (a) and (b)**.
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Important properties of Fourier series
Continuous-time sinusoidal signals: Periodicity

$\cos(\pi t)$

$\cos(2\pi t)$

$\cos(4\pi t)$
Continuous-time complex exponentials: Periodicity

\[ e^{j\pi t} \quad t = \frac{3}{4} \quad e^{j2\pi t} \]
Discrete-time sinusoidal and complex exponentials: Periodicity

(In DT the angular frequency is $\Omega$, whereas in CT it is $\omega$, sorry!).

Consider the DT sinusoidal signal $x_2[n] = \cos(j\Omega n)$. This signal is:

(a) Always periodic.
(b) Never periodic.
(c) Periodic for some values of $\Omega$.

Consider the DT complex exponential $x_1[n] = e^{j\Omega n}$. This signal is:

(a) Always periodic.
(b) Never periodic.
(c) Periodic for some values of $\Omega$. 

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Consider the discrete time complex exponentials $x[n] = e^{j\Omega n}$. If $x[n]$ is periodic with period $N$ then

$$x[n] = x[n + N]$$

Not every value of $\Omega$ produces a periodic signal. If we assume that $x[n]$ is periodic,

$$x[n + N] = e^{j\Omega(n+N)}$$
$$= e^{j\Omega n} e^{j\Omega N}$$
$$= x[n]$$

hence we need that $\Omega N = 2\pi k \rightarrow \Omega = 2\pi \frac{k}{N}$. Similarly, sinusoidal signals are periodic if and only if $\Omega = 2\pi \frac{k}{N}$.
Discrete-time sinusoidal signals

\[ \Omega = 2\pi \left( \frac{1}{32} \right) \quad N = 32 \]

\[ \Omega = 2\pi \left( \frac{2}{32} \right) \quad N = 16 \]

\[ \Omega = 2\pi \left( \frac{4}{32} \right) \quad N = 8 \]
Discrete-time sinusoidal signals

\[ \Omega = 2\pi \left( \frac{8}{32} \right) \quad N = 4 \]

\[ \Omega = 2\pi \left( \frac{16}{32} \right) \quad N = 2 \]

\[ \Omega = 2\pi \left( \frac{32}{32} \right) \quad N = 1 \]
Discrete-time sinusoidal signals

\[
\Omega = \frac{2\pi(32 + 1)}{32} \quad N = 32
\]

\[
\Omega = \frac{2\pi(32 + 2)}{32} \quad N = 16
\]

\[
\Omega = \frac{2\pi(32 + 4)}{32} \quad N = 8
\]
Discrete-time sinusoidal signals

We have found *some sinusoidal signals with different frequencies* that are identical. Why is that?
Discrete-time complex exponentials

\[ \Omega = 2\pi \left( \frac{1}{32} \right) \]
\[ \Omega = 2\pi \left( \frac{2}{32} \right) \]
\[ \Omega = 2\pi \left( \frac{4}{32} \right) \]
\[ \Omega = 2\pi \left( \frac{8}{32} \right) \]
\[ \Omega = 2\pi \left( \frac{16}{32} \right) \]
\[ \Omega = 2\pi \left( \frac{32}{32} \right) \]
Discrete-time complex exponentials

As in the case with the sinusoidal signals, we have found some complex exponentials with different frequencies that are identical.
Consider the discrete time complex exponentials $x_1[n] = e^{j\Omega_1 n}$ and $x_2[n] = e^{j\Omega_2 n}$, where $\Omega_2 = \Omega_1 + 2\pi$. The angular frequency of $x_2[n]$ is higher than the angular frequency of $x_1[n]$, $\Omega_2 > \Omega_1$.

However, $x_1[n]$ and $x_2[n]$ are the same signal:

$$
x_2[n] = e^{j\Omega_2 n} = e^{j(\Omega_1 + 2\pi) n} = e^{j\Omega_1 n} e^{j2\pi n} = e^{j\Omega_1 n} = x_1[n]
$$

In discrete-time, all the possible complex exponentials that can be generated are within any interval of frequencies of size $2\pi$, for instance $[-\pi, \pi]$. 

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Discrete-time complex exponentials

We have seen that

- In order for a complex exponential to be periodic, its angular frequency $\Omega$ must be such that $\Omega = 2\pi \frac{k}{N}$, where $N$ is the period (whenever $k$ and $N$ have no factors in common).

- The frequencies $\Omega_1$ and $\Omega_2 = \Omega_1 + 2\pi$ produce the same signal, since they visit the same points in the complex plane.

We can conclude that there only exist $N$ different complex exponentials of period $N$, namely

$$0, \ 2\pi \frac{1}{N}, \ 2\pi \frac{2}{N}, \ \ldots, \ 2\pi \frac{N-1}{N}$$

For instance, $\Omega = 2\pi \frac{2N+2}{N}$ produces the same signal as $\Omega = 2\pi \frac{2}{N}$ and $\Omega = -2\pi \frac{2}{N}$ produces the same signal as $\Omega = 2\pi \frac{N-2}{N}$
### Summary

<table>
<thead>
<tr>
<th>CT complex exponentials</th>
<th>DT complex exponentials</th>
</tr>
</thead>
<tbody>
<tr>
<td>Always periodic</td>
<td>Only periodic for $\Omega = \frac{2\pi k}{N}$, $k, N$ integers</td>
</tr>
<tr>
<td>Different frequencies produce different signals</td>
<td>Frequencies within an interval of size $2\pi$ produce different signals</td>
</tr>
<tr>
<td>There exist infinite complex exponentials with period $T$, namely those of frequencies $\frac{2\pi}{T}, 2\frac{2\pi}{T}, 3\frac{2\pi}{T}, \ldots$</td>
<td>There only exist $N$ complex exponentials with period $N$, namely those of frequencies $\frac{2\pi}{N}, 2\frac{2\pi}{N}, \ldots, N\frac{2\pi}{N}$</td>
</tr>
</tbody>
</table>
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Quick review

The notion of frequency in discrete-time signals

Fourier series representation of discrete-time periodic signals

Important properties of Fourier series
Fourier series in CT

A Fourier series is a representation of a periodic signal as a linear combination of harmonically related complex exponentials. By harmonically related we mean that their frequencies can be expressed as an integer multiple of the fundamental frequency.

For instance, in continuous-time, a periodic signal $x_T(t)$ with period $T$ can be expressed as

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk \frac{2\pi}{T} t}$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency, $k\omega_0$ are its harmonics and $a_k$ are its coefficients.
What’s different in discrete-time? Just the fact that there are only $N$ different complex exponentials with period $N$!

Hence, the Fourier series representation of a periodic discrete-time signal $x_N[n]$ with period $N$ is

$$x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N} n}$$

where $\Omega_0 = 2\pi/N$ is the fundamental frequency, $k\Omega_0$ are its harmonic frequencies and $a_k$ are its coefficients.

This equation is, of course, a synthesis equation.
How do we obtain the coefficients of the Fourier series of a discrete-time periodic signal $x_N[n]$?

One approach would be to solve the following system of $N$ equations and $N$ unknowns ($a_k$):

\[
\begin{align*}
    x_N[0] &= \sum_{k=\{N\}} a_k \\
    x_N[1] &= \sum_{k=\{N\}} a_k e^{jk\Omega_0} \\
    x_N[2] &= \sum_{k=\{N\}} a_k e^{jk\Omega_02} \\
    \cdots \\
    x_N[N - 1] &= \sum_{k=\{N\}} a_k e^{jk\Omega_0(N-1)}
\end{align*}
\]
Fourier series: Determining the coefficients I

In matrix form, the resulting system of linear equations is:

$$
\begin{bmatrix}
  x_N[0] \\
  x_N[1] \\
  x_N[2] \\
  \vdots \\
  x_N[N-1]
\end{bmatrix}
= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\
  1 & e^{j\Omega_0} & e^{j2\Omega_0} & \cdots & e^{j(N-1)\Omega_0} \\
  1 & e^{j\Omega_02} & e^{j2\Omega_02} & \cdots & e^{j(N-1)\Omega_02} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & e^{j\Omega_0(N-1)} & e^{j2\Omega_0(N-1)} & \cdots & e^{j(N-1)\Omega_0(N-1)}
\end{bmatrix}
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  \vdots \\
  a_{N-1}
\end{bmatrix}
$$
Another option is using an **analysis equation**. Analysis equations use the fact that harmonically related exponentials are **orthogonal**, so that

$$\sum_{n=\{N\}} e^{j k_1 \Omega_0 n} (e^{j k_2 \Omega_0 n})^* = \sum_{n=\{N\}} e^{j k_1 \Omega_0 n} e^{-j k_2 \Omega_0 n}$$

$$= \sum_{n=\{N\}} e^{j (k_1 - k_2) \Omega_0 n}$$

$$= \begin{cases} N & \text{if } k_1 - k_2 = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$
Fourier series: Determining the coefficients II

So we know that

\[
x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n},
\]

\[
\sum_{n=\langle N \rangle} e^{jk_1\Omega_0 n} (e^{jk_2\Omega_0 n})^* = \begin{cases} N & \text{if } k_1 - k_2 = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}
\]

Let us calculate

\[
\sum_{n=\langle N \rangle} x_N[n](e^{jm\Omega_0 n})^* = \sum_{n=\langle N \rangle} \left[ \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \right] e^{-jm\Omega_0 n} = a_m N
\]

Hence

\[
a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x_N[n] e^{-jk\Omega_0 n}
\]
Fourier series of periodic signals: summary

**Continuous-time**, $\omega_0 = \frac{2\pi}{T}$

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x_T(t) e^{-jk\omega_0 t} dt$$

**Analysis**

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

**Synthesis**

**Discrete-time**, $\Omega_0 = \frac{2\pi}{N}$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x_N[n] e^{-jk\Omega_0 n}$$

$$x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n}$$

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Parseval’s relation

The average power of a periodic signal $x_N[n]$ can be calculated both in the time domain and by using the coefficients of its Fourier series:

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

The coefficient $a_k$ of the Fourier series of $x_N[n]$ tells us how much of the harmonic frequency $k\omega_0$ there is in the signal.
Linearity and time shifting

Consider two periodic signals $x_N[n]$ and $y_N[n]$ with period $N$ and Fourier coefficients $a_k$ and $b_k$, respectively. Then:

- The signal $z_N[n] = A x_N[n] + B y_N[n]$ is periodic with period $N$ and its Fourier coefficients are $c_k = Aa_k + Bb_k$.

- The signal $v_N[n] = x_N[n - n_0]$ is periodic with period $N$ and its Fourier coefficients are $d_k = e^{jk \Omega_0 n_0} a_k$. 

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Fourier series and LTI systems

\[ x_N(n) \rightarrow H(\Omega) \rightarrow y_N(n) \]

\[ x_N[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \rightarrow y_N[n] = \sum_{k=\langle N \rangle} H(k\Omega_0) a_k e^{jk\Omega_0 n} \]

Hence, the Fourier coefficients of \( y_N[n] \) are \( b_k = a_k H(k\Omega_0) \). In words, they are a filtered version of the coefficients \( a_k \).