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Introduction

The frequency domain: what for?

The convolution theorem

Introduction to filters
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The convolution theorem

Introduction to filters
What is this week about?

The main topics covered by this course are organised as follows:

Week 1: CT and DT signals and systems in the time domain.
Week 2: CT signals and systems in the frequency domain.
Week 3: DT signals and systems in the frequency domain.
Week 4: Sampling theory and communication systems.
What have we learnt so far?

1. CT and DT **signals in the time domain**: basic signals, representation, properties, classification, manipulations in the time domain (shift, reflection, amplification) . . .
2. CT and DT **systems in the time domain**: properties, LTI systems, impulse response, convolution . . .
3. CT **signals in the frequency domain**: Fourier series and Fourier transform.
The notion of frequency: sinusoidal signals

Consider the signals

\[ x(t) = 2 \cos(2\pi t) \]
\[ y(t) = 2 \cos(4\pi t) \]

(a) The frequency of \( x(t) \) is \textbf{higher} than the frequency of \( y(t) \).
(b) The frequency of \( x(t) \) is \textbf{lower} than the frequency of \( y(t) \).
(c) The frequencies of \( x(t) \) and \( y(t) \) are \textbf{equal}.
The notion of frequency: sinusoidal signals

\[ x(t) = 2\cos(2\pi t) \]

\[ y(t) = 2\cos(4\pi t) \]
The notion of frequency: complex exponentials

Consider the signals

\[ x(t) = e^{j2\pi t} \]
\[ y(t) = e^{-j2\pi t} \]

(a) The period of \( x(t) \) is \( T_x = 1 \) and the period of \( y(t) \) is \( T_y = -1 \).
(b) The period of \( x(t) \) is \( T_x = -1 \) and the period of \( y(t) \) is \( T_y = 1 \).
(c) The periods of \( x(t) \) and \( y(t) \) are \( T_x = T_y = 1 \).
The notion of frequency: complex exponentials

Note that the sum of the complex exponentials $e^{j2\pi t}$ and $e^{-j2\pi t}$ results in a real signal! That’s why the magnitude of the FT of real signals is always symmetric :)
The frequency domain and the Fourier transform

\[ x(t) \overset{FT}{\leftrightarrow} X(f), X(\omega) \]

The \( f \)-domain

\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \]  
Analysis

\[ x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \]  
Synthesis

The \( \omega \)-domain

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \]
Consider the Fourier transforms:

(a) $X(f)$ has low frequencies and $Y(f)$ high frequencies.
(b) $X(f)$ has high frequencies and $Y(f)$ low frequencies.
(c) $Y(f)$ has low and high frequencies, $X(f)$ neither of them.
Properties of FT: temporal displacement

Consider the FT pairs:

\[
\begin{align*}
x(t) \quad \overset{FT}{\Longleftrightarrow} & \quad X(f) \\
y(t) \quad \overset{FT}{\Longleftrightarrow} & \quad Y(f)
\end{align*}
\]

If \( y(t) = x(t - 10) \), then

(a) \( |Y(f)| = |X(f - 10)| \).
(b) \( |Y(f)| = |X(f - 0.1)| \).
(c) \( |Y(f)| = |X(f)| \).
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Digital equalisers
Applications

- By looking at the **frequency domain we can extract useful information**. Why? Because many natural phenomena are **cyclic** (electromagnetic radiation, movement of planets, circadian rhythms...).
- It can be easier to **understand signal distortions caused by physical media in the frequency domain**. Why? Because media can often be described as linear and time-invariant.
- **Modulation techniques for transmitting data** can be best understood in the frequency domain.
- **Signal processing techniques** can be best understood in the frequency domain.
Internet of Things

IoT devices:

- **Measure and process physical signals**, and information might be more apparent in the frequency domain.
- **Digitise physical signals**, and the process of digitisation can be best understood in the frequency domain.
- **Transmit information** by using modulation techniques and they can be understood in the frequency domain.
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Linear Time-Invariant systems

LTI systems are defined by two basic properties:

1. **Linearity**: Combinations of inputs produce combinations of their outputs.
2. **Time invariance**: Delayed inputs produce delayed outputs.

\[
x(t) \xrightarrow{LTI \text{ system}} y(t)
\]

\[
x_1(t) \rightarrow y_1(t) \quad x_2(t) \rightarrow y_2(t)
\]

\[
A_1 x_1(t) + A_2 x_2(t) \rightarrow A_1 y_1(t) + A_2 y_2(t)
\]

\[
x_1(t - t_0) \rightarrow y_1(t - t_0)
\]

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Linear Time-Invariant systems and complex exponentials

A pure frequency $\omega$ at the input produces the same pure frequency $\omega$ at the output (with different amplitude and phase):

$$e^{j\omega_0 t} \xrightarrow{} H_0 e^{j\omega_0 t}$$

$$e^{j\omega_1 t} \xrightarrow{} H_1 e^{j\omega_1 t}$$

$$\vdots$$

$$e^{j\omega t} \xrightarrow{} H(\omega) e^{j\omega t}$$

$$A_0 e^{j\omega_0 t} + A_1 e^{j\omega_1 t} \xrightarrow{}$$

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Linear Time-Invariant systems and general signals

A pure frequency $\omega$ at the input produces the same pure frequency $\omega$ at the output (with different amplitude and phase):

\[ e^{j\omega_0 t} \rightarrow H_0 e^{j\omega_0 t} \]
\[ e^{j\omega_1 t} \rightarrow H_1 e^{j\omega_1 t} \]
\[ \vdots \]
\[ e^{j\omega t} \rightarrow H(\omega) e^{j\omega t} \]

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \]
Linear Time-Invariant systems and general signals

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \longrightarrow \quad y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t} d\omega \]

\[ y(t) \overset{\text{FT}}{\longleftrightarrow} Y(\omega) = X(\omega) H(\omega) \]

\( H(\omega) \) is the **frequency response** or **transfer function** of the LTI system.
Linear Time-Invariant systems: Summary

\[ x(t), X(\omega) \rightarrow \text{LTI system} \rightarrow y(t), Y(\omega) \]

\[ y(t) = x(t) \ast h(t) \iff Y(\omega) = X(\omega)H(\omega) \]

Is there any relationship between \( h(t) \) and \( H(\omega) \)? Can you guess?
The Convolution Theorem

**Question:** What is the Fourier transform of a convolution?

Consider $y(t) = x(t) * h(t)$. Let us calculate its Fourier transform.

\[
Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt
\]

\[
= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \right] e^{-j\omega t} dt
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(t - \tau) e^{-j\omega(t-\tau)} e^{-j\omega\tau} d\tau dt
\]

\[
= \int_{-\infty}^{\infty} x(\tau) \left[ \int_{-\infty}^{\infty} h(t - \tau) e^{-j\omega(t-\tau)} dt \right] e^{-j\omega\tau} d\tau
\]

\[
= \int_{-\infty}^{\infty} x(\tau) H(\omega) e^{-j\omega\tau} d\tau
\]

\[
= X(\omega) H(\omega)
\]

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Linear Time-Invariant systems: Summary

\[ x(t), \, X(\omega) \rightarrow H(\omega) \rightarrow y(t), \, Y(\omega) \]

\[ y(t) = x(t) \ast h(t) \quad \overset{FT}{\iff} \quad Y(\omega) = X(\omega)H(\omega) \]

\[ \begin{align*}
x(t) & \quad \overset{FT}{\iff} \quad X(\omega) \\
h(t) & \quad \overset{FT}{\iff} \quad H(\omega) \\
y(t) & \quad \overset{FT}{\iff} \quad Y(\omega) \\
\end{align*} \]
What does the frequency response tell us?

\[ x(t), X(\omega) \xrightarrow{H(\omega)} y(t), Y(\omega) \]
What does the frequency response tell us?
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LTI systems as filters

The frequency response $H(\omega)$ shows that LTI systems act as **frequency filters** since they allow certain frequencies at the input to pass whereas they stop other frequencies.

\[
Y(\omega) = X(\omega)H(\omega)
\]
LTI systems as filters

The frequency response of LTI systems are characterised by

- The **stopband**: interval of frequencies that are not allowed to pass.
- The **passband**: interval of frequencies that are allowed to pass.
- A **bandwidth**: width of the passband (ONLY POSITIVE FREQUENCIES ARE CONSIDERED).

There are three basic types of filters:

- **Lowpass** filters: Low frequencies pass.
- **Highpass** filters: High frequencies pass.
- **Bandpass** filters: Frequencies within an intermediate band pass.
Ideal filters

\[ H_{LP}(\omega) \]

Lowpass filter:
\[ H_{LP}(\omega) \]
\[ B_{LP} = \omega_H - \omega_L = \omega_H \]

\[ H_{HP}(\omega) \]

Highpass filter:
\[ H_{HP}(\omega) \]
\[ B_{HP} = \infty - \omega_L = \infty \]

\[ H_{BP}(\omega) \]

Bandpass filter:
\[ H_{BP}(\omega) \]
\[ B_{BP} = \omega_H - \omega_L \]

\( \omega_L \): lower cutoff, \( \omega_H \): upper cutoff, \( \omega_C \): centre frequency
Example: Filtering complex exponentials

\[ e^{j\omega_0 t} \rightarrow H(\omega) \rightarrow y(t) \]
Example: Filtering sinusoidal signals
Example: Filtering periodic signals

\[ x_T(t) \rightarrow H(\omega) \rightarrow y(t) \]

\[ x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T} \]
The inverse filter

An LTI system with frequency response $H_2(\omega)$ is said to be the inverse filter of another LTI system with frequency response $H_1(\omega)$ if

$$H_2(\omega) = \frac{1}{H_1(\omega)}$$

In this example,

$$Z(\omega) = Y(\omega)H_2(\omega) = X(\omega)H_1(\omega)H_2(\omega) = X(\omega)H_1(\omega)\frac{1}{H_1(\omega)} = X(\omega)$$

as long as $H_1(\omega) \neq 0$. 

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Remember!

- In continuous-time, the higher the frequency of a signal, the faster its amplitude changes.
- A frequency-domain description of a signal tells us how fast the amplitude of the signal changes: it can be fast, slow or even both!
- LTI systems can also be described in the frequency domain: the frequency response tells us which frequencies the system attenuates and which frequencies it amplifies.

Never forget this:

\[
y(t) = x(t) \ast h(t) \iff Y(\omega) = X(\omega)H(\omega)
\]