Marking and Feedback

It is essential to your success in this module that you complete all the coursework. For each coursework there are 10 possible marks to be earned. A portion of these marks will be assigned for completing the assignment with clear evidence of effort (4-5 marks for complete worked solutions to all exercises). A portion of the coursework will be marked in more detail. This marking provides more detailed feedback on these sections and emulates exam marking of scripts. This portion is worth 5-6 marks out of 10. These solutions may be discussed at subsequent exercise classes and additional feedback provided.

References

- Ch. 2 Fundamental Planetary Science, Lissauer and de Pater
- Ch. 1 Physical Processes in the Solar System, Landstreet
- Ch. 2 Introductory Astronomy & Astrophysics (hereafter IAA), by Zeilik, Gregory and Smith (several copies in the library)
- Ch. 2 & 3 Moons & Planets, Hartmann.
- Ch. 2 An Introduction to Modern Astrophysics, by Carroll and Ostlie (there are several copies in the library).
- Note: the Planet & Satellite Calculator website, by Doug Hamilton, has many parameters pre-loaded and may be useful for doing the calculations. http://janus.astro.umd.edu/AW/awtools.html#calculators

Please do not do very repetitive calculations by hand. Ask if you have questions.

Useful Information

- \( G = 6.67408 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \)
- 1 Astronomical Unit (AU) = 1.496 \times 10^{11} \text{ metres}
- The Sun’s radius \( R_\odot = 6.96 \times 10^8 \text{ metres} \)

<table>
<thead>
<tr>
<th>Object</th>
<th>Mass (kg)</th>
<th>Radius (km)</th>
<th>Rotation Period</th>
<th>Semi-major axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>1.98 \times 10^{30}</td>
<td>6.96 \times 10^3</td>
<td>~ 25 days</td>
<td></td>
</tr>
<tr>
<td>Jupiter</td>
<td>1.90 \times 10^{27}</td>
<td>71,398</td>
<td>9.92 h</td>
<td>5.203 AU</td>
</tr>
<tr>
<td>Saturn</td>
<td>5.68 \times 10^{26}</td>
<td>60,330</td>
<td>10.66 h</td>
<td>9.537 AU</td>
</tr>
<tr>
<td>Neptune</td>
<td>1.028 \times 10^{26}</td>
<td>24,764</td>
<td>16.1 h</td>
<td>30.06 AU</td>
</tr>
</tbody>
</table>
Exercises

1. The Vis-viva Equation
   a) Derive an expression for the total mechanical energy of a body of mass \( m \) orbiting a much larger mass \( M \) with a circular orbit of radius \( a \). Express your answer in terms of masses and orbital elements.

   Now consider the orbit of \( m \) to be eccentric orbit with semi-major axis \( a \). Derive the following expression for the speed of the object at a distance \( r \) from \( M \).

   \[
   v^2 = GM \left(\frac{2}{r} - \frac{1}{a}\right)
   \]  
   (1)

   b) Is this relation valid for eccentric orbits as well? Why? or Why not?

   c) In September and October of 2017, a new minor planet was discovered moving through the inner solar system. This object was observed at a heliocentric distance of 0.45 AU moving with a speed of 68.1 km/s. Assuming the object has a mass of \( m = 3.5 \times 10^9 \) kg, compute the mechanical energy of this object with respect to the Sun. Using this value, comment on this object’s provenance (i.e., where it formed/originated) with respect to the solar system.

2. Orbital angular momentum
   a) Use the vis-viva equation and the expression for the pericenter to write the orbital angular momentum \( L \) of an orbit in terms of the masses, semi-major axis \( (a) \) and eccentricity \( (e) \).

   b) How does angular momentum scale with orbital semi-major axis (i.e. \( L \propto a^n \) what is \( n \))?

3. Angular momentum in the solar system. The Astronomy Workshop Calculator may be useful here.
   a) Treating the Sun as a sphere of uniform density, calculate its rotational angular momentum. Recall that the expression for rotational angular momentum is

   \[
   I_{\text{spin}} = I \Omega
   \]  
   (2)

   where \( I = KmR^2 \) is the moment of inertia (with \( K = 2/5 \) for a uniform sphere) and \( \Omega = 2\pi/P_{\text{rot}} \) is the constant rotational angular velocity.

   b) Compute Jupiter’s orbital angular momentum

   c) Which has greater orbital angular momentum

      i. Jupiter or Saturn?

      ii. Saturn or Neptune?

   d) Using the Astronomy Workshop planetary calculator, calculate the ratio between a planet’s spin angular momentum and orbital angular momentum.

   e) Where is most of the solar system’s angular momentum? Is it in planetary spins or orbits? Which body/bodies are accountable for most of the solar system’s angular momentum?
4. The synchronous or co-rotation orbit plays an important role in the tidal evolution of satellites and planets. We will discuss tides and their implications later in the module.

   a) Derive a general expression for the semi-major axis of a planetary satellite whose orbital period about the planet is equal to the planet's rotation period. This is the so-called synchronous orbit \( (a_c) \).

   b) Compute the semi-major axis of the synchronous orbit for each of the terrestrial and giant planets. Express your answer in both meters and planetary radii (again the Astronomy Workshop website might be useful here).

   c) Compute synchronous radius of the Sun, listing your answer in terms of solar radii and AU. How does this compare with Mercury's orbit? Is it comparable in size? Is it much much smaller?

5. Resonances

   Resonance occurs whenever a system is driven or forced at one of its natural frequencies. Orbital resonances may occur when orbits interact with an integer multiple of their natural frequencies. For example, a so-called 'mean motion' orbital resonance occurs when two
bodies have orbital periods that are a simple integer ratio of each other. These resonances are labelled by the integers defining this ratio \((k : j)\) where \(k\) and \(j\) are integers. For example the 3:7 mean motion resonance occurs when the orbital periods \((P_i)\) of two objects roughly satisfy \(3P_1 = 7P_2\).

Figure 1 shows the eccentricity of objects discovered near and beyond Neptune’s orbit. There are four resonances near Neptune that appear populated with Kuiper belt objects.

a) Does the orbital distribution of Kuiper Belt Objects appear uniform and randomly distributed?

b) Determine the semi-major axes of the one or two populated resonant locations from the figure and compute the orbital periods. Compare these orbital periods with that of Neptune and determine the integer ratios of the populated exterior resonances of Neptune (i.e., what are \(k : j\) for these resonances?).

6. Planning spacecraft missions (Transfer Orbits)

This problem examines some issues related to getting to Mercury \((a = 0.3871\ \text{AU})\) from Earth \((a = 1.0000\ \text{AU})\). For simplicity consider that both Earth and Mercury on circular orbits. After launch the spacecraft is on a circular orbit about the Sun with a semi-major axis equal to the Earth’s \((a = 1.0000\ \text{AU})\) and is no longer ‘near’ the Earth in its orbit (i.e. you may now ignore gravitational interaction with Earth). It will need to ‘do a burn’ to change its speed to reach Mercury’s orbit. Treat the burn as an instantaneous change in speed or \(\Delta v\).

a) The new, ‘post burn’ transfer orbit must take the spacecraft from Earth’s orbit to Mercury’s (i.e. it must cross both). Make a diagram showing Earth’s and Mercury’s orbit and identify the pericentre and apocentre of the transfer orbit (consider the orbit that requires a minimum of energy change - this is tangent to, or just grazes, both Earth’s and Mercury’s circular orbits).

b) Compute the semi-major axis and eccentricity of the transfer orbit.

c) After doing a burn, the spacecraft will be at the apocentre of the transfer orbit. Compute the speed of the spacecraft just after the burn.

d) Compute the spacecraft’s circular orbital velocity when at 1AU before the burn and the \(\Delta v\) needed to go from a a circular orbit with semi-major axis of 1AU to the transfer orbit.

e) How long does it take for the spacecraft to travel from Earth’s orbit to Mercury’s orbit (i.e., from apocentre to pericentre of the transfer orbit)?

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1(Here the 'mean motion' refers to the mean or average orbital angular velocity \(\omega = (GM/a^3)^{1/2} = 2\pi / P_{\text{orbit}}\). This is a mean value over an orbit as the angular velocity is greater at pericentre and lesser at apocentre.

2As it turns out the orbital structure of the Kuiper belt is an important key to deciphering how the giant planets formed and evolved. Understanding the structure of this distribution is a principal aim of planetary science at the moment and would make a good subject for a 3rd year project.