EBU4375 Signals and Systems Theory

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Basic Time Signals

Basic Continuous-Time Signals

• The Unit-Step Function
• The Unit-Impulse Function
• Complex Exponential and Sinusoidal Signals

Basic Discrete-Time Signals

• The Unit-Step Sequence
• The Unit-Impulse Sequence
• Complex Exponential and sinusoidal Sequence
The Unit-Step Function (CT Signals)

• the unit (or Heaviside) step function is defined as

\[ u(t) = \begin{cases} 
1 & t > 0 \\
0 & t < 0 
\end{cases} \]

i.e. it’s discontinuous at \( t = 0 \)

• the shifted (retarded) step function is similarly defined as

\[ u(t - t_0) = \begin{cases} 
1 & t > t_0 \\
0 & t < t_0 
\end{cases} \]

\( t_0 = 1 \)
The unit-impulse (Dirac-delta) function is defined as

\[
\delta(t) = \begin{cases} 
0 & t \neq 0 \\
\infty & t = 0
\end{cases} \quad \text{for} \quad t = 0 
\implies \int_{-\varepsilon}^{\varepsilon} dt \delta(t) = 1
\]

\varepsilon \to 0
The Unit-Impulse Function (CT Signals)

• It is also defined by

\[
\int_{a}^{b} dt \phi(t) \delta(t) = \begin{cases} 
\phi(0) & a < 0 < b \\
0 & a < b < 0 \text{ or } 0 < a < b \\
\text{undefined} & a = 0 \text{ or } b = 0
\end{cases}
\]

• A delayed (retarded) delta function \(\delta(t - \tau)\) is defined by

\[
\int_{-\infty}^{\infty} dt \phi(t) \delta(t - \tau) = \phi(\tau)
\]

(1)
The Unit-Impulse Function (CT Signals)

Properties of $\delta(t)$:

$$\delta(at) = \frac{1}{|a|} \delta(t)$$

$$\delta(-t) = \delta(t) \quad (2)$$

$$x(t)\delta(t) = x(0)\delta(t) \quad \text{(if } x(t) \text{ is continuous at } t = 0)$$

$$x(t)\delta(t - \tau) = x(\tau)\delta(t - \tau) \quad \text{(if } x(t) \text{ is continuous at } t = \tau)$$

A continuous-time signal $x(t)$ may be expressed as (we prove this in the following lecture)

$$x(t) = \int_{-\infty}^{\infty} d\tau \, x(\tau)\delta(t - \tau)$$
Complex Exponential and Sinusoidal (CT Signals)

Euler’s formula: \(e^{jw_0t} = \cos(w_0t) + j \sin(w_0t)\)

where \(j = \sqrt{-1}\), \(w_0 \neq 0\) is real, and \(t\) is the time.
Complex Exponential and Sinusoidal (CT Signals)

Since

\[ e^{jw_0(t + \frac{2\pi}{|w_0|})} = e^{jw_0t} e^{j2\pi \frac{w_0}{|w_0|}} = e^{jw_0t} e^{j2\pi \text{sign}(w_0)} = e^{jw_0t} \]

we have

\[ e^{jw_0t} \text{ is periodic with fundamental period } \frac{2\pi}{|w_0|} \]

Note that

\[ e^{j2\pi k} = 1, \text{ for } k = 0, \pm 1, \pm 2, \ldots. \]
Complex Exponential and Sinusoidal (CT Signals)

- $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period
- Energy in $e^{j\omega_0 t}$: $\int_{-\infty}^{\infty} |e^{j\omega_0 t}|^2 dt = \int_{-\infty}^{\infty} 1 dt = \infty$
- Average Power in $e^{j\omega_0 t}$: $\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |e^{j\omega_0 t}|^2 dt$
  
  $= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} 1 dt = 1$
Complex Exponential and Sinusoidal (CT Signals)

\[ Ce^{at} \]

where \( C \) and \( a \) are complex numbers.

If

\[ C = |C|e^{j\theta} \quad \text{and} \quad a = r + j\omega_0 \]

then

\[ Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(w_0t+\theta)} \]

\[ = |C|e^{rt}\cos(w_0t + \theta) + j|C|e^{rt}\sin(w_0t + \theta) \]

\[ \text{Re}(Ce^{at}) + j\text{Im}(Ce^{at}) \]
Complex Exponential and Sinusoidal (CT Signals)
Periodicity and Fundamental Period (CT Signals)

Consider two sinusoidal functions \( x(t) = \sin(\omega_0 t + \theta) \) and \( x_m(t) = \sin(m\omega_0 t + \theta) \). The fundamental angular frequencies of these two CT signals are given by \( \omega_0 \) and \( m\omega_0 \) radians/s, respectively. In other words, the angular frequency of the signal \( x_m(t) \) is \( m \) times the angular frequency of the signal \( x(t) \). In such cases, the CT signal \( x_m(t) \) is referred to as the \( m \)th harmonic of \( x(t) \).

Examples of harmonics.
(a) Waveform for the sinusoidal signal \( x(t) = \sin(2\pi t) \); (b) waveform for its second harmonic given by \( x_2(t) = \sin(4\pi t) \).
Periodicity and Fundamental Period (CT Signals)

**Proposition**  A signal $g(t)$ that is a linear combination of two periodic signals, $x_1(t)$ with fundamental period $T_1$ and $x_2(t)$ with fundamental period $T_2$ as follows:

$$g(t) = ax_1(t) + bx_2(t)$$

is periodic iff

$$\frac{T_1}{T_2} = \frac{m}{n} = \text{rational number}.$$

The fundamental period of $g(t)$ is given by $nT_1 = mT_2$ provided that the values of $m$ and $n$ are chosen such that the greatest common divisor (gcd) between $m$ and $n$ is 1.
Periodicity and Fundamental Period (CT Signals)

Example
Determine if the following signals are periodic. If yes, determine the fundamental period.
\[ g_1(t) = 3 \sin(4\pi t) + 7 \cos(3\pi t); \]

Solution
\[ \frac{T_1}{T_2} = \frac{1/2}{2/3} = \frac{3}{4} \]

fundamental period of \( g_1(t) \) is given by \( nT_1 = 4T_1 = 2 \text{ s} \).

fundamental period of \( g_1(t) \) can also be evaluated from \( mT_2 = 3T_2 = 2 \text{ s} \).
The Unit-Step Sequence (DT Signals)

- The unit-step sequence $u[n]$ is defined by

$$u[n] = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0 
\end{cases} \quad (3)$$

Unlike $u(t)$, $u[n]$ is defined at $n = 0$

- The shifted unit-step sequence $u[n-k]$ is similarly defined by

$$u[n-k] = \begin{cases} 
1 & n \geq k \\
0 & n < k 
\end{cases} \quad (4)$$
The Unit-Impulse Sequence (DT Signals)

- The unit-impulse (or unit-sample) sequence $\delta[n]$ is defined by
  \[ \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \]  \hspace{1cm} (5)

- The shifted unit-impulse (sample) sequence $\delta[n-k]$ is similarly defined by
  \[ \delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases} \]  \hspace{1cm} (6)
The Unit-Impulse Sequence (DT Signals)

• Unlike $\delta(t)$, $\delta[n]$ is readily defined. From (5) and (6) it is evident that

\[
x[n]\delta[n] = x[0]\delta[n] \\
x[n]\delta[n-k] = x[k]\delta[n-k]
\]

are the discrete-time counterparts of

\[
x(t)\delta(t) = x(0)\delta(t) \\
x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)
\]

• from (3) and (4), $\delta[n]$ and $u[n]$ are related by

\[
\delta[n] = u[n] - u[n-1] \\
u[n] = \sum_{k=0}^{\infty} \delta[n-k]
\]

A discrete-time signal $x[n]$ may be expressed as (we prove this in the following lecture)

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]
\]
Complex Exponential and Sinusoidal (DT Signals)

\[ x[n] = e^{jw_0 n} \]

\( e^{jw_0 n} \) is periodic \iff \( e^{jw_0 n} = e^{jw_0 (n+M)} \) for some integer \( M > 0 \)

\( \iff e^{jw_0 M} = 1 \) for some integer \( M > 0 \)

\( \iff w_0 M = 2\pi m \) for some integers \( m, M > 0 \)

\( \iff \frac{w_0}{2\pi} \) is rational.
Complex Exponential and Sinusoidal (DT Signals)

- If $\frac{w_0}{2\pi} = \frac{m}{M}$ for some integers $m$ and $M$ which have no common factors, then the fundamental period is

$$M = \frac{2m\pi}{w_0}$$

because

$$e^{jw_0(n+N)} = e^{jw_0n}e^{j\frac{2\pi m}{M}N}$$
Periodicity and Fundamental Period (DT Signals)

Examples

1) Is $x[n] = e^{jn2\pi/3} + e^{jn3\pi/4}$ periodic? If it is periodic, what’s its fundamental period?

   For $e^{jn2\pi/3}$, $w_0/(2\pi) = 1/3$, so $e^{jn2\pi/3}$ is periodic with fundamental period 3.

   For $e^{jn3\pi/4}$, $w_0/(2\pi) = 3/8$, so $e^{jn3\pi/4}$ is periodic with fundamental period 8.

   $x[n]$ is periodic with fundamental period $24 = lcm(3, 8)$. 
Periodicity and Fundamental Period (DT Signals)

Examples

2) Is $x[n] = \sin \left( \frac{3n}{4} \right)$ periodic? If it is periodic, what’s its fundamental period?

Since $\frac{\omega_0}{2\pi} = \frac{3}{8\pi}$ is irrational, $x[n]$ is not periodic
Periodicity and Fundamental Period (DT Signals)

Examples

3) Is \( x[n] = \sin(8\pi n/31) \) periodic? If it is periodic, what’s its fundamental period?

Since \( w_0/(2\pi) = 4/31 \), \( x[n] \) is periodic with fundamental period 31

\[
x[0] = x[31] = 0
\]

Note that the continuous-time signal \( \sin(8\pi t/31) \) has fundamental period 31/4

But \( x[n] \) has no \( 31/4 \)-th sample and it misses 0 between \( x[7] \) and \( x[8] \)
Operations – Amplitude Scaling (CT Signals)

Given a CT signal $x(t)$, scaling consist of multiplying it by a scalar value $a$, producing the new signal $y(t) = ax(t)$.

![Graph of x(t) and 2x(t)]

Scaling is defined in an analogous way for DT signals.
Operations – Time Shift (CT Signals)

Given a CT signal $x(t)$, time shifting by $t_0$ units of time produces the new signal $y(t) = x(t - t_0)$ (DT shifting is defined in a similar way).
Operations – Time Inversion/Reversal (CT Signals)

Time reversal *flips* the time axis producing the signal $y(t) = x(-t)$. 

![Graph showing time reversal](image-url)
Operations – Time Scaling (CT Signals)

Time scaling **expands** or **compresses** the time axis. Signal $y(t) = x(at)$ is a compressed version of $x(t)$ if $|a| > 1$, and an expanded version if $|a| < 1$. 
Consider the signal $x(t) = u(-2t + 3)$. In order to obtain $x(t)$ we will take the following steps:

- Define the time-shift $y(t) = u(t + 3)$.
- Define the time scaling and reverse $z(t) = y(-2t)$.

As we can see, $z(t) = y(-2t) = u(-2t + 3)$ and therefore $x(t) = z(t)$. 
Operations – Combined Time Operations (CT Signals)

In general, we can obtain the signal $y(t) = x(-at + t_0)$ by shifting $x(t)$ first and then by scaling and time reversing the result. Graphically, the signal $u(-2t + 3)$ can be obtained as follows:

You can see that $u(-2(1.5) + 3) = u(0)$. 
Operations – Sum (CT Signals)

Adding two signals $x(t)$ and $y(t)$ means adding their values each time instant.
Operations – Product (CT Signals)

Similarly, we multiply signals by multiplying their values each time instant.
Operations – Time Shift (DT Signals)

When a DT signal $x[k]$ is shifted by $m$ time units, the delayed signal $\phi[k]$ is expressed as

$$\phi[k] = x[k + m]$$

If $m < 0$, the signal is said to be delayed in time.
Operations – Time Shift (DT Signals)

Example
Represent the DT sequence shown in (a) as a function of time-shifted DT unit impulse functions.

\[ x[k] = \delta[k+1] + 2\delta[k] + 3\delta[k-1] \]
Operations – Time Inversion/Reversal (DT Signals)

\[ y(n) = x(-n) \]

positive time switches to negative time and vice versa

**Example**

Sketch the time-inverted version of the following DT sequence:

\[ x[k] = \begin{cases} 
1 & -4 \leq k \leq -1 \\
0.25k & 0 \leq k \leq 4 \\
0 & \text{elsewhere,}
\end{cases} \]
Operations – Time Inversion/Reversal (DT Signals)

**Solution**
To derive the expression for the time-inverted signal $x[-k]$, substitute $k = -m$

$$x[-m] = \begin{cases} 
1 & -4 \leq -m \leq -1 \\
-0.25m & 0 \leq -m \leq 4 \\
0 & \text{elsewhere.}
\end{cases}$$

Simplifying the above expression and expressing it in terms of the independent variable $k$ yields

$$x[-m] = \begin{cases} 
1 & 1 \leq m \leq 4 \\
-0.25m & -4 \leq m \leq 0 \\
0 & \text{elsewhere.}
\end{cases}$$
Operations – Time Inversion/Reversal (DT Signals)

(a) Original CT sequence $x[k]$
(b) Time-inverted version $x[-k]$
Operations – Time Scaling (DT Signals) also known as Decimation and Interpolation

(a) Original DT sequence $x[k]$.  
(b) Decimated version $x[2k]$, of $x[k]$.  
(c) Interpolated version $x[0.5k]$ of signal $x[k]$. 

(a) 

(b) 

(c)
Operations – Combined Time Operations (DT Signals)

Example
Sketch the waveform for $x[-15 - 3k]$ for the DT sequence $x[k]$
Solution
Express \( x[-15 - 3k] = x[-3(k + 5)] \) and follow steps (i)–(iii) as outlined below.

(i) Compress \( x[k] \) by a factor of 3 to obtain \( x[3k] \).

(ii) Time-reverse \( x[3k] \) to obtain \( x[-3k] \).

(iii) Shift \( x[-3k] \) towards the left-hand side by five time units to obtain \( x[-3(k + 5)] = x[-15 - 3k] \).
Operations – Combined Time Operations (DT Signals)

(a) Original DT signal $x[k]$.
(b) Time-scaled version $x[3k]$.
(c) Time-inverted version $x[-3k]$ of (b).
(d) Time-shifted version $x[-15-3k]$ of (c).