RELATIVISTIC INVARIANCE OF THE S-MATRIX

We wish to prove the important statement that the S-matrix is relativistic invariant.

To this end, we begin by noticing that $d^4 x$ and $L_i(x)$ are relativistic invariant - the only thing we need to look at is the instruction of Time-ordering: What happens to $T(L_i(x_1)...L_i(x_n))$ when we perform a Lorentz transformation?

Consider, for instance, the 2nd order term

$$S^{(2)} = \frac{i^2}{2} \int d^4x_1 d^4x_2 \left[ L_i(x_1) L_i(x_2) \right]$$

Consider also a frame with origin at $x_2$.

$x_2$ could lie in various places:

1) If it falls within the LC of $x_1$, so if $(x_2 - x_1)^2 > 0$ (TIMELIKE SEPARATION), then:

Here are no problems. Recall that the Lorentz transformation does not change the sign of a timelike vector - hence if $(x_2 - x_1)^2 > 0$, the time ordering between $x_2$ and $x_1$ will not be changed (what ever it is) by a Lorentz transformation.
b) However, if \( (x_2 - x_1)^2 < 0 \) (space-like separation), then we need to worry.

However, if \( (x_1 - x_2)^2 < 0 \), then operators must commute. This is because of the principle of microcausality:

Since \( (x_1 - x_2)^2 < 0 \), no light ray can start at \( x_2 \) and reach \( x_1 \) - hence any such \( x_2 \) (i.e. any \( x_2 \) outside the 2 light cones at \( x_1 \)) is causally disconnected from \( x_1 \).

This implies that measures of observables at \( x_1 \) and \( x_2 \) cannot influence each other if \( (x_1 - x_2)^2 < 0 \), as this would need a signal propagating faster than the speed of light.

As a consequence:

\[
[\mathcal{O}(x_1), \mathcal{O}(x_2)] = 0 \quad \text{for} \quad (x_1 - x_2)^2 < 0
\]

A local operator \( \mathcal{O}(x_1) \). In our case, \( \mathcal{O}(x_1) \) and \( \mathcal{O}(x_2) \) commute for \( x_1 \) and \( x_2 \) space-like separated.
To summarise:

a) For $(x_1 - x_2)^2 > 0$, the ordering of the times is a Lorentz invariant quantity.

b) For $(x_1 - x_2)^2 < 0$, the ordering of the times is not Lorentz invariant, however the fields commute.

Hence $T(\mathcal{L}_1(x_1), \mathcal{L}_2(x_2))$ is Lorentz invariant.

This can be generalised to all T-products in the Dyson expansion.
WICK THEOREM

From the Dyson expansion we know that
we have to compute objects of the kind

$$T \left( \phi_I(x_1) \cdots \phi_I(x_n) \right)$$

where \( \phi_I(x) \equiv :\phi(x)… : \) is a

normal ordered product of fields at the same

space time point. For instance in QED

$$\phi_I(x) = - e \phi(x) \phi_A(x) \phi(x) :$$

Wick theorem gives us a way to compute

such expressions.

We begin with a simpler case where we have

just \(+\) products of many fields (without the

extra complication of normal ordering).

Let’s begin by looking at the quickly \( \phi(x) \phi(y) \).

Using \( \phi = \phi^+ + \phi^- \) we have

\[
\phi(x) \phi(y) = \phi(x) \phi(y) + \phi(x) \phi(y) + \phi(y) \phi(x) + \phi(y) \phi(x)
\]

all of them are normal ordered

\[\text{this is not}\]

\[\phi(x) \phi(y) = \phi(x) \phi(y) + \phi(y) \phi(x) - \phi(y) \phi(x)\]

where I reordered the term conflicting with

the normal ordering term by adding (antimultiply \( :\phi ; \))

and multiplying \( \phi^- (y) \phi^+(x) \).
Hence \[
\phi(x)\phi(y) = :\phi(x)\phi(y): + i\phi(x)\phi(y) + \Gamma\phi(x)\phi(y) : \Gamma
\]

Recall that
\[
[\phi(x), \phi(y)] = i \Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip(x-y)}}{2E(p)}
\]

Let us now have
\[
\phi(y)\phi(x) = :\phi(y)\phi(x): + \phi(y)\phi(x) = \min \quad :\phi(y)\phi(x): = :\phi(x)\phi(y):
\]

\[
= :\phi(x)\phi(y): + i \Delta^{(+)}(y-x) =
\]

\[
= :\phi(x)\phi(y): - i \Delta^{(-)}(x-y) \quad \Rightarrow
\]

\[
\phi(y)\phi(x) = :\phi(x)\phi(y): - i \Delta^{(-)}(x-y)
\]

We can then write
\[
T(\phi(x)\phi(y)) = \Theta(x_0-y_0)( :\phi(x)\phi(y): + i \Delta^{(+)}(x-y)) + \Theta(y_0-x_0)( :\phi(x)\phi(y): - i \Delta^{(-)}(x-y))
\]

\[
= :\phi(x)\phi(y): + i \Delta(x-y)\Theta(x_0-y_0) - i \Delta(x-y)\Theta(y_0-x_0)
\]

\[
\text{THE FEYNMAN PROPAGATOR}
\]

\[
= :\phi(x)\phi(y): + i \Delta(x-y)
\]
Hence we conclude that

\[ T(\phi(x) \phi(y)) = \phi(x) \phi(y) + i\Delta F(x-y) \]

Finally the Feynman propagator enters the scene.

Recall also that \( i\Delta F(x-y) = \langle 0 | T(\phi(x) \phi(y)) | 0 \rangle \)

The equation above is laconic: that with this. Indeed, taking the \( \langle 0 | 10 \rangle \) one gets

\[ \langle 0 | T(\phi(x) \phi(y)) | 0 \rangle = \langle 0 | \phi(x) \phi(y) | 0 \rangle + i\Delta F(x-y) \]

\[ = 0 \text{ by construction} \]

which is indeed an identity.

---

**Generalisation to Many Fields**

Without proof (but check it at home for 3 fields) we write down the extension to many fields of Wick's theorem, see MS p. 96.

We use the notation

\[ A(x) B(y) = \langle 0 | T(A(x) B(y)) | 0 \rangle \]

When "---" is called **contraction**

Here are the contractions for the various cases:
REAL KG: \( \phi(x) \phi(y) = \lambda \Delta \phi(x-y) \)

COMPLEX KG: \( \phi(x) \phi(y) = \lambda \Delta \phi(x-y) \)

DIRAC: \( \psi(x) \psi(y) = \lambda \Delta \phi(x-y) \)

In this notation,

\[
T(ABC \ldots Z) = : AB \ldots Z : + \\
+ : AB \ldots CD \ldots Z : + : A B C D \ldots E \ldots Z : + \ldots + : A B \ldots YZ : + \\
+ : A B \ldots C D \ldots E \ldots F \ldots Z : + \ldots + : A B \ldots WX YZ : + \\
+ \text{terms with 3 contractions etc etc}
\]

Note: the \( \square \) contraction is a
C-number and can be taken out of the :

IMPORTANT CAVEAT: if the fields to take out
are bosonic no problem. If they are fermionic one
has to include a SIGN of

\[
(\text{-})
\]

in front of this term.
FOUR-FIELD EXAMPLE:

We have established that

\[ T(\phi_1 \phi_2) = : \phi_1 \phi_2 : + \phi_1 \phi_2 \]

Now, for 4-points we have

Terms with 0 \( \phi \):

\[ T(\phi_1 \cdots \phi_4) = : \phi_1 \phi_2 \phi_3 \phi_4 : + \]

Terms with 1 \( \phi \):

\[ + \phi_1 \phi_2 : \phi_3 \phi_4 : + \phi_1 \phi_3 : \phi_2 \phi_4 : + \phi_1 \phi_4 : \phi_2 \phi_3 : + \]

Terms with 2 \( \phi \):

\[ + \phi_2 \phi_3 : \phi_1 \phi_4 : + \phi_2 \phi_4 : \phi_1 \phi_3 : + \phi_3 \phi_4 : \phi_1 \phi_2 : + \]

Terms with 3 \( \phi \):

\[ + \phi_3 \phi_4 : \phi_1 \phi_2 : + \phi_4 \phi_3 : \phi_1 \phi_2 : + \phi_4 \phi_2 : \phi_1 \phi_3 : + \phi_2 \phi_4 : \phi_1 \phi_3 : \]
GENERALISATION OF WICK'S THEOREM TO $T$-PRODUCTS OF $::$

Wick gave us the extension of his theorem to $T$-products of normal ordered products which is what we need to deal with.

Let's first say it in words: we have the same formula as before except that one has to omit any contraction between fields belonging to the same $::$.

Here is the proof. Inside every $::$, we give a positive shift $+\varepsilon_0$ to all the $\phi(-)$ parts (which in the $::$ have to be on the left) and a negative time shift $-\varepsilon_0$ to all the $\phi(+)$. So for instance we have, focusing on one block,

$$\phi_1(x) \phi_2(x) \phi_3(x) \cdots = \phi_1(-) \phi_2(-) \phi_3(+) \cdots \phi_m(+)$$

Yet $x \to x + \varepsilon_0$ and $x \to x_0 - \varepsilon_0$.

For this block, the $T$-ordering coincides with the normal ordering since we do not need to reorder fields. There is no contraction arising from inside the same normal-ordered term.
INTRODUCTION TO FEYNMAN DIAGRAMS

To begin our study of the Feynman diagram representation of scattering amplitudes we will consider two simple theories of scalar fields, namely the "$\phi^4$" and "$\phi^3$" theories.

1ST ORDER TERM IN $\phi^4$ THEORY: 4-PT SCATTERING

We consider the theory defined by

$$ L = L_0 + L_{\text{int}} \quad \text{with} $$

$$ L_0 = \frac{1}{2} (\partial \phi)(\partial \phi) - \frac{1}{2} m^2 \phi^2, \quad L_{\text{int}} = -\frac{g}{4!} \phi^4 $$

With this theory we wish to compute to lowest order the 4-PT scattering amplitude.

\[ 1+2 \to 1' + 2' \]

Initial states

Final states

More precisely, we take as initial state

$$ |i\rangle = |\,i\,\rangle = a(\hat{p}_1)^\dagger a(\hat{p}_2)^\dagger |0\rangle $$

$$ |f\rangle = a(\hat{p}_3)^\dagger a(\hat{p}_4)^\dagger |0\rangle $$

We will expand $\phi$ as
\[\phi(x) = \int d^3\rho \, N(\rho) \left[ a(\rho)^e + a^*(\rho)^e \right]\]

Using the Dyson expansion,

\[S_{fi} = \langle f | S | i \rangle = \lim_{t_0 \to -\infty} \langle f | U(t, t_0) \, | i \rangle = \]

\[= \langle f | T \, \exp \left( i \int_{t_0}^u dx : \phi^e : \right) \, | i \rangle\]

with \( | f \rangle \) and \( | i \rangle \) specified earlier.

We will look up to first order, where then

\[S = 11 + i \left( -\frac{A}{4!} \right) \int d^4 x : \phi^e : + O(x^2)\]

Note how this is a perturbative expansion with

Also, note that in this first order term we do not need to write the T matrix.

Then we have

\[S_{fi} = \langle f | 1 - \frac{2}{4!} \int d^4 x : \phi : | i \rangle + O(x^2) = \]

\[= \langle 0 | a(\vec{p}_3) a(\vec{p}_4) \, a(\vec{p}_1) a(\vec{p}_2) | 0 \rangle + \]

\[- \frac{i}{4!} \langle 0 | a(\vec{p}_3) a(\vec{p}_4) \int d^4 x : \phi(x) : a(\vec{p}_1) a(\vec{p}_2) | 0 \rangle\]
The first term (the zeroth order term)

\[ \langle 0 | a(p_3) a(p_4) a(p_1) a(p_2) | 0 \rangle = 0 \]

has nothing to do with the interaction (there is no $\lambda$), it just describes the free propagation of the particles.

We can evaluate it as follows:

\[ Z = \langle 0 | a(p_3) [a(p_4), a(p_1)] a(p_2) | 0 \rangle + \]
\[ + \langle 0 | a(p_3) a(p_1) a(p_4) a(p_2) | 0 \rangle = \]

\[ = \frac{1}{(2\pi)^3} \int d^3 p_4 \int d^3 p_1 \langle 0 | a(p_3) a(p_2) | 0 \rangle + \]

\[ + \frac{1}{(2\pi)^3} \int d^3 p_3 \int d^3 p_1 \langle 0 | a(p_3) a(p_2) | 0 \rangle = \]

\[ = \frac{1}{(2\pi)^3} \int d^3 p_4 \int d^3 p_1 \delta(p_4 - p_1) \langle 0 | a(p_3) a(p_2) | 0 \rangle + \]

\[ + \frac{1}{(2\pi)^3} \int d^3 p_3 \int d^3 p_1 \delta(p_3 - p_1) \langle 0 | a(p_4) a(p_2) | 0 \rangle = \]

\[ = \frac{1}{(2\pi)^3} \int d^3 p_4 \int d^3 p_1 \delta(p_4 - p_1) \delta(p_3 - p_1) + \]

\[ + \frac{1}{(2\pi)^3} \int d^3 p_3 \int d^3 p_1 \delta(p_3 - p_1) \delta(p_4 - p_1) \]

\[ \langle 0 | a(p_3) a(p_4) a(p_1) a(p_2) | 0 \rangle = \frac{1}{(2\pi)^3} \int d^3 p_1 \delta(p_3 - p_1) \delta(p_4 - p_1) \]

\[ + \delta(p_3 - p_2) \delta(p_4 - p_1) \]
This describes the two possibilities:

\[
\begin{align*}
\frac{s_0}{4!} & \rightarrow \frac{p_1}{4i} \rightarrow \frac{p_3}{4i} + \frac{p_2}{4i} \rightarrow \frac{p_4}{4i} \\
\frac{s}{4} & \rightarrow \frac{p_4}{4i} \rightarrow \frac{p_2}{4i} \rightarrow \frac{p_3}{4i}
\end{align*}
\]

This is just the \textit{FREE SCATTERING} of the two particles, without the interaction.

**FIRST-ORDER TERM**

Next we look at the first-order term. This is given by:

\[
S_1^{(1)} = \left\langle p_3, p_4 \right| \left( - \frac{i}{4!} \right) \int d^4 x \ T \left( \phi (x) \phi (y) \phi (z) \phi (w) \right) \left| p_1, p_2 \right\rangle =
\]

using (formally) Wick's theorem

\[
= \left\langle p_3, p_4 \right| \left( - \frac{i}{4!} \right) \int d^4 x : \phi (x) \phi (y) \phi (z) \phi (w) : \left| p_1, p_2 \right\rangle =
\]

Hence we wish to compute

\[
\int \frac{d^4 x}{4!} \left\langle 0 \left| a (p_3) a (p_4) : \phi (x) \phi (y) \phi (z) \phi (w) : a (p_1) a (p_2) \right| 0 \right\rangle
\]

Recall that

\[
\phi (x) = \int d^3 p \ N (p) \left[ e^{ip \cdot x} \frac{a (p)}{\phi (-)} + e^{-ip \cdot x} \frac{a (p) \phi (p)}{\phi (+)} \right]
\]
We are dealing with 2 $\Phi^+$'s and 2 $\Phi^-$'s in our states, hence out of the $\Phi^4$ operator we need to extract the term which can survive. This is going to be the term with $2\Phi^+ \text{ AND } 2\Phi^-$. The reason for this is that the # of $\Phi^+$ and $\Phi^-$ must be the same otherwise I will get something like

$$\langle n_1 n_2 \rangle = 0 \quad \text{for} \ n\neq m$$

Such otherwise the interaction must destroy the initial state and create the final state.

Next we realize that

$$\Phi^4 = \langle \Phi^+\Phi^-\Phi^+\Phi^-\Phi^+\Phi^-\Phi^+\Phi^- \rangle =$$

$$= \binom{4}{2} \left( \Phi^+ \right)^2 \left( \Phi^- \right)^2 + \text{terms which don't have } 2\Phi^+\text{'s and } 2\Phi^-\text{'s}$$

$$= \binom{4}{2} \left( \Phi^+ \right)^2 \left( \Phi^- \right)^2$$

The "4 choose 2" factor is because out of the 4 $\Phi$'s we choose 2 to be $\Phi^+$ and the remaining ones will have necessarily to be $\Phi^-$. Obviously, the possible combinations are

$$(12), (13)(14), (23)(24), (34)$$
Next compute

\[ \langle 0 | a(q_1) a(q_2) \left[ a(p) a(p') \right]^\dagger \phi(q_1)(q_2) x_{q_1 q_2} \phi^\dagger(p)(p') | 0 \rangle = \]

\[ = \int dK_1 dK_2 dq_1 dq_2 e^{i(K_1+K_2-q_1-q_2)x} \frac{N(K_1)N(K_2)N(q_1)N(q_2)}{N} \]

\[ \langle 0 | a(q_1) a(q_2) a(K) a(K^\dagger q_1 q_2 a(q_2) a(p_1) a(p_2) | 0 \rangle \]

Focus on

\[ \langle 0 | a(q_1) a(q_2) a(p_1) a(p_2) | 0 \rangle = \]

\[ = a(q_1) [ a(q_2) a(p_1) a^\dagger(p_2) | 0 \rangle + \]

\[ + a(q_1) a^\dagger(p_1) a(q_2) a^\dagger(p_2) | 0 \rangle \]

\[ \quad \text{for the nophon with } [ \ ] \]

The specific value of [ ] depends on our choice of \( N \). In our earlier choice, when

\[ N = \frac{1}{(2\pi)^3 V} \]

\[ \text{then } [a(p) a^\dagger(q)] = \sqrt{\frac{1}{(2\pi)^3}} \delta(p-q) \]

while for \( N = \frac{4}{(2\pi)^3 \sqrt{V}} \)

\[ \text{then } [a(p) a^\dagger(q)] = \frac{1}{(2\pi)^3} \delta(p-q) \]

Let's use the former. Then we get
\[ \phi = (2\pi)^3 (2\omega(p_1)) \delta(p_1 - q_2) \ a(q_1) \ a(p_2) \ 0 \rangle + \\
\]

\[ + \ a(q_1) \ a(p_1) \ (2\pi)^3 \delta(q_1 - p_2) \ (2\omega(p_2)) \ 0 \rangle = \\
\]

\[ = (2\pi)^6 (2\omega(p_1))(2\omega(p_2)) \left[ \delta(q_2 - p_1) \delta(q_1 - p_2) + \\
+ \delta(q_2 - p_2) \delta(q_1 - p_1) \right] \ 0 \rangle \\
\]

\[ \Rightarrow \ a(q_1) \ a(q_2) \ a^\dagger(p_1) \ a^\dagger(p_2) \ 0 \rangle = \\
\]

\[ = (2\pi)^6 (2\omega(p_1))(2\omega(p_2)) \left[ \delta(q_2 - p_1) \delta(q_1 - p_2) + \\
+ \delta(q_2 - p_2) \delta(q_1 - p_1) \right] \\
\]

Similarly

\[ \langle 0 \mid a(p_3) \ a(p_4) \ a(k_1) \ a(k_2) \rangle = \\
\]

\[ = (2\pi)^6 (2\omega(p_3))(2\omega(p_4)) \left[ \delta(k_1 - p_4) \delta(k_2 - p_3) + \\
+ \delta(k_1 - p_3) \delta(k_2 - p_4) \right] \\
\]

Hence we find
\[ \langle 0 | a(p_3) a(p_4) (\phi(x)) (\phi(x)) a(p_1) a(p_2) | 0 \rangle = \]

\[ = \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \]

\[ \times e^{i(k_1+k_2-q_1-q_2) \cdot x} \times \frac{1}{(2\pi)^3 (2\pi | p_1 \rangle (2\pi | p_2 \rangle) (2\pi | p_3 \rangle (2\pi | p_4 \rangle) - \frac{\delta(k_1-p_3) \delta(k_2-p_4) + \delta(k_1-p_4) \delta(k_2-p_3)}{8} \]

\[ + \frac{\delta(q_4-p_1) \delta(q_2-p_3) + \delta(q_4-p_2) \delta(q_2-p_1)}{8} \]

\[ = 4 \pi \]

Thus we get

\[ \langle 1 | \frac{1}{4!} \int d^4 x \frac{T(\phi(x))}{\chi(\phi(x))} p_1 p_2 \rangle = \]

\[ = -i \frac{4!}{4^4} \cdot \frac{1}{(2\pi)^4} \int d^4 x e^{i(p_3+p_4-p_1-p_2) \cdot x} \]

\[ = -i \chi \frac{4!}{4^4} \cdot \frac{1}{(2\pi)^4} \delta(p_1 + p_2 - p_3 - p_4) \]

\[ = -i \chi \frac{4!}{4^4} \cdot \frac{1}{(2\pi)^4} \delta(p_1 + p_2 - p_3 - p_4) \]

Note how the x integration enforces momentum conservation.
This could be obtained using the following pictorial Feynman rules.

1. For each external line attach a factor of 1.
   (This 1 is because we have scalars.)

2. For each vertex (interactions are usually called vertices, short for interaction vertices) put a factor of \(-i\chi\).

3. For each vertex put a factor of
   \[
   (2\pi)^4 \delta \left( \sum_{i} p_i - \sum_{f} p_f \right)
   \]
   which ensures momentum conservation.

We still need to see how to glue vertices together using propagators. In order to do this, let's take a theory with simpler couplings.