MSci/MSc Examination by Course Unit

Tuesday 29th April 2014 14:30 - 17:00

PHY7004U/P Relativistic Waves and Quantum Fields Duration: 2 hours 30 minutes

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Instructions:

Answer ONLY THREE questions. Each question carries 20 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Only non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Answer ONLY THREE of the five questions (a FORMULA SHEET is provided at the end of the paper)

**Question 1: The Lorentz group**

(a) Consider the coordinate transformation \( x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \). Write down under what conditions the matrix \( \Lambda \) represents a Lorentz transformation. Show that the matrix

\[
L(\beta) := \begin{pmatrix}
\gamma & 0 & 0 & -\gamma \beta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\gamma \beta & 0 & 0 & \gamma \\
\end{pmatrix}
\]

represents a Lorentz transformation, namely a boost along the \( \hat{z} \)-axis. Here \( \gamma := \frac{1}{\sqrt{1 - \beta^2}} \).

[4 marks]

(b) Consider a four-vector \( x^\mu \), with \( x^2 > 0 \), i.e. \( x \) is timelike. Show that it is always possible to find a frame where the coordinates of \( x \) are of the form \((x^0', \vec{0})\) and determine the Lorentz transformation relating the initial frame to this particular frame.

[5 marks]

(c) Under a Lorentz transformation \( \Lambda \), the Dirac field \( \psi(x) \) transforms as

\[
\psi(x) \rightarrow \psi'(x') := S(\Lambda)\psi(x),
\]

where the \( 4 \times 4 \) matrix \( S(\Lambda) \) satisfies \( S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda) = \Lambda^{\mu}_{\nu}\gamma^{\nu} \). Find how the quantity

\[
G^{\mu\nu}(x) := i\bar{\psi}(x)[\gamma^{\mu}, \gamma^{\nu}]\psi(x)
\]

transforms under a Lorentz transformation. You may use without proof that \( \gamma^0S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda) \).

[5 marks]

(d) The rapidity \( y \) of a particle of energy \( E \) moving along the \( \hat{z} \)-axis with momentum \( p_z \) is defined as

\[
y := \frac{1}{2} \log \frac{E + p_z}{E - p_z}.
\]

Show that, under the particular Lorentz transformation \( L(\beta) \) introduced in part (a), the rapidity transforms as

\[
y \rightarrow y' = y + \frac{1}{2} \log \frac{1 - \beta}{1 + \beta}.
\]

Find also the approximate form of this transformation when \( \beta \ll 1 \), i.e. the transformation to first order in \( \beta \). Next, consider two boosts \( L(\beta_1) \), \( L(\beta_2) \) along the \( \hat{z} \)-axis with parameters \( \beta_1 \) and \( \beta_2 \). Show that under the Lorentz transformation \( L(\beta_2) \circ L(\beta_1) \) obtained by performing the two boosts \( L(\beta_1) \) and \( L(\beta_2) \) in a sequence, the rapidity transforms in an additive way, that is

\[
y \rightarrow y' = y + \frac{1}{2} \log \frac{1 - \beta_1}{1 + \beta_1} + \frac{1}{2} \log \frac{1 - \beta_2}{1 + \beta_2}.
\]

[6 marks]
Question 2: Quantisation of the complex scalar field

(a) The Lagrangian density describing the dynamics of a complex scalar field $\phi$ is given by

$$L = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi.$$ 

Write down the expression of the conjugate momenta to $\phi$ and $\phi^\dagger$, and the Euler-Lagrange equations for the fields. Write down the expression of the Hamiltonian density.

[4 marks]

(b) Consider the phase transformation for the field $\phi$

$$\phi \rightarrow \phi' = e^{i\alpha} \phi.$$ 

Write down the transformation for $\phi^\dagger$ and prove that the Lagrangian is invariant under these transformations. Write down the expression of the corresponding Noether current.

[4 marks]

(c) Use the free-field expansion

$$\phi(x) = \int d^3 p N(p) \left[ a(p)e^{-ip\cdot x} + b^\dagger(p)e^{ip\cdot x} \right],$$

with the normalisation $N(p) := 1/[(2\pi)^3 2E(p)]$, to show that the momentum operator

$$P^i := \int d^3 x \left[ (i\partial^0 \phi)^\dagger (\partial^i \phi) + (\partial^i \phi)^\dagger (i\partial^0 \phi) \right]$$

can be written as

$$P^i = \int \frac{d^3 p}{(2\pi)^3 2E(p)} p^i \left[ a^\dagger(p)a(p) + b(p)b^\dagger(p) \right].$$

[6 marks]

(d) Use the expression of $P^i$ in terms of the oscillators given in part (c) to prove that

$$[P^i, \phi(x)] = i \nabla^i \phi(x).$$

You may use without proof the commutation relations between the oscillators

$$\begin{align*}
[a(p), a^\dagger(q)] &= [b(p), b^\dagger(q)] = (2\pi)^3 2E(p) \delta^{(3)}(p - q), \\
[a(p), a(q)] &= [b(p), b(q)] = [a(p), b(q)] = 0.
\end{align*}$$

[6 marks]
**Question 3:** Quantisation of the Dirac field

(a) The Lagrangian for a massive Dirac field is given by \( \mathcal{L} = \overline{\psi} (i \gamma^\mu \partial_\mu - m) \psi \), where \( \overline{\psi} = \psi^\dagger \gamma^0 \). Derive the Euler-Lagrange equations for the fields. Find the conjugate momenta to \( \psi \) and \( \overline{\psi} \), and derive the Hamiltonian density \( H \). Show that the classical Hamiltonian density can be written as \( H = i \psi^\dagger \partial_0 \psi \) upon using the equations of motion. [4 marks]

(b) The free-field expansion of the Dirac field in terms of oscillators has the form

\[
\psi(x) = \sum_{r=1,2} \int d^3p \, N(p) \left[ a_r(p) u_r(p) e^{-ip \cdot x} + b_r^\dagger(p) v_r(p) e^{ip \cdot x} \right],
\]

with the normalisation \( N(p) = 1/[(2\pi)^3 \sqrt{2E(p)}] \). Using the expression of the Hamiltonian density given in part (a), prove that the normal-ordered Hamiltonian \( H := \int d^3x : i \psi^\dagger \partial_0 \psi : \) in terms of the oscillators is given by

\[
H = \int \frac{d^3p}{(2\pi)^3} E(p) \sum_{r=1,2} \left[ a_r^\dagger(p) a_r(p) + b_r^\dagger(p) b_r(p) \right].
\]

In this derivation, you may use without proof the following identities:

\[
u_r^\dagger(p) u_s(p) = v_r^\dagger(p) v_s(p) = 2E(p) \delta_{rs}, \quad u_r^\dagger(p) v_s(p) = v_r^\dagger(p) u_s(p) = 0.
\]

[6 marks]

(c) Prove that the canonical commutation relations have the form

\[
\{ \psi_i(x, t), \psi_j^\dagger(y, t) \} = \delta_{ij} \delta^{(3)}(x - y),
\]

where \( i = 1, \ldots, 4 \), labels the various components of the Dirac spinor. Using the canonical commutation relations given in part (c), prove that

\[
i[H, \psi(x)] = \partial_0 \psi(x).
\]

For this question you can omit the normal-ordering prescription in the definition of \( H \), i.e. you can use \( H = \int d^3x i \psi^\dagger \partial_0 \psi \). [5 marks]

(d) Consider a massless Dirac fermion, with Lagrangian \( \mathcal{L}_{m=0} = i \overline{\psi} \gamma^\mu \partial_\mu \psi \), and the transformation on \( \psi \) defined by \( \delta \psi = i \alpha \gamma^5 \psi \), where \( \alpha \) is a constant parameter. Find the transformation of \( \overline{\psi} \), and prove that the \( \mathcal{L}_{m=0} \) is invariant under this transformation. Using the equations of motion, prove that \( \partial_\mu J^\mu = 0 \), where \( J^\mu = \overline{\psi} \gamma^\mu \gamma^5 \psi \). You may use without proof that \( \{ \gamma^\mu, \gamma^5 \} = 0 \). [5 marks]
Question 4: Noether’s theorem and dilatations

(a) Consider the theory of a massless fermion $\psi$, described by the Dirac Lagrangian
\[ L_0 = i \bar{\psi} \gamma^\mu \partial_\mu \psi, \]
where $\bar{\psi} = \psi^\dagger \gamma^0$, and consider the global phase transformations $\psi \rightarrow \psi' = e^{i\alpha} \psi$, with $\alpha$ being a constant parameter. Find the transformation of $\bar{\psi}$, and prove that global phase transformations are a symmetry of $L_0$. Write down the expression of the corresponding Noether current.

(b) Prove that the corresponding action $S = \int d^4 x \ L_0$, where $L_0$ is given in part (a), is invariant under dilatations, which are defined as
\[ x' = e^{-\lambda} x, \quad \psi'(x') = e^{\frac{3}{2} \lambda} \psi(x). \]
For this question you may find it more convenient to work with finite transformations.

(c) Dilatations are an example of transformations whose Jacobian $|J|$ is not equal to one. Consider then a generic theory with Lagrangian density $L(\varphi, \partial_\mu \varphi)$ describing the dynamics of the field $\varphi(x)$, and let $x^\mu \rightarrow x^\mu + \delta x^\mu$ be a coordinate transformation. Assume that $\varphi$ transforms as
\[ \varphi(x) \rightarrow \varphi'(x') := \varphi(x) + \delta_T \varphi(x), \quad \text{where} \quad \delta_T \varphi(x) := \delta \varphi(x) + \delta x^\mu (\partial_\mu \varphi)(x), \]
and further assume that the action $S := \int d^4 x \ L(\varphi, \partial_\mu \varphi)$ is invariant under this transformation, i.e.
\[ \int d^4 x' L(\varphi'(x'), \partial'_\mu \varphi'(x')) = \int d^4 x L(\varphi(x), \partial_\mu \varphi(x)). \]
For the case of a transformation with $|J| \neq 1$, prove that there is a conserved Noether current, whose expression is
\[ \delta J^\mu = \frac{\partial L}{\partial (\partial_\mu \varphi)} \delta \varphi + L \delta x^\mu. \]
You may use without proof the fact that the Jacobian is $|J| := \det_{\mu, \nu} \left( \frac{\partial x'^\nu}{\partial x^\mu} \right) \simeq 1 + \partial_\mu \delta x^\mu$.

(d) Finally, use this result to determine the Noether current associated to dilatations (introduced in part (b)) for the theory of a massless fermion with Lagrangian density $L_0$ defined in part (a).
Question 5: Propagators and the S-matrix

(a) Consider the field theory of a real scalar \( \phi \), described by the Lagrangian density

\[
L_0 = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2} \phi^2 ,
\]

and introduce the \( T \)-product of two free fields

\[
T(\phi(x)\phi(0)) := \theta(x_0)\phi(x)\phi(0) + \theta(-x_0)\phi(0)\phi(x).
\]

Show with a direct calculation that

\[
(\Box + m^2)T(\phi(x)\phi(0)) = -i\delta^{(4)}(x) ,
\]

i.e. by acting with the \( \Box + m^2 \) operator on the expression \( \theta(x_0)\phi(x)\phi(0) + \theta(-x_0)\phi(0)\phi(x) \). You may use without proof the equal-time commutation relations:

\[
[\phi(x, t), \phi(y, t)] = 0, \quad \text{and} \quad [\phi(x, t), \partial_t \phi(y, t)] = i\delta^{(3)}(x - y).
\]

(b) Use the free-field expansion

\[
\phi(x) = \int d^3 p \, N(p) \left[ a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x} \right],
\]

with the normalisation \( N(p) := 1/[((2\pi)^3/2E(p))] \), together with the canonical commutation relations

\[
[a(p), a^\dagger(q)] = (2\pi)^3/2E(p)\delta^{(3)}(p - q), \quad [a(p), a(q)] = 0,
\]

to show that

\[
\langle 0|\phi(x)\phi(y)|0 \rangle = i\Delta^{(+)}(x - y),
\]

where

\[
i\Delta^{(+)}(x) := \int \frac{d^3 k}{(2\pi)^3} \frac{e^{-iE(k)t + ik\cdot x}}{2E(k)}.
\]

(c) Consider now adding to \( L_0 \) an interaction term of the form

\[
L_1 = \frac{\lambda}{3!} :\phi^3: ,
\]

so that the Lagrangian density of the theory is now \( L = L_0 + L_1 \), where \( L_0 \) is given in part (a). Find the dimension of the coupling constant \( \lambda \), and write down the Dyson expansion for the \( S \)-matrix of the theory.

(d) Consider the initial state \( |i\rangle := a^\dagger(p_1)|0\rangle \) decaying into the final state \( |f\rangle = a^\dagger(p_2)a^\dagger(p_3)|0\rangle \). Determine the matrix element \( \langle f|S|i \rangle \) to the first order in \( \lambda \).
Formula Sheet: (in units where $\hbar = c = 1$)

- Four-vectors:
  
  \[ a \cdot b = a^\mu b_\mu = a_\mu b^\mu = a_\mu b^\nu \eta_{\mu\nu} = a_\mu b_\nu \eta^{\mu\nu} \quad \text{with} \quad \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

  \[ x^\mu = (t, \vec{x}), \quad x_\mu = (t, -\vec{x}), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \begin{pmatrix} \partial/\partial t, -\vec{\nabla} \end{pmatrix}, \quad \partial_\mu = \partial/\partial x^\mu = \begin{pmatrix} \partial/\partial t, \vec{\nabla} \end{pmatrix}, \quad \hat{p}^\mu = i\partial^\mu, \quad \hat{p}_\mu = i\partial_\mu. \]

- Klein-Gordon equation:
  
  \[ (-\hat{p} \cdot \hat{p} + m^2)\phi = (\partial_\mu \partial^\mu + m^2)\phi = (\Box + m^2)\phi = 0. \]

- Free Dirac equation in Hamiltonian form:
  
  \[ i\frac{\partial}{\partial t}\psi = (\vec{\alpha} \cdot \hat{p} + \beta m)\psi = (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)\psi, \quad \text{or in covariant form:} \]

  \[ (i\partial - m)\psi = (i\gamma^\mu \partial_\mu - m)\psi = (\hat{p} - m)\psi = (\gamma \cdot \hat{p} - m)\psi = (\gamma^\mu \hat{p}_\mu - m)\psi = 0. \]

- Dirac and Gamma matrices:
  
  \[ (\alpha^i)^2 = \mathbb{I}, \quad i = 1, 2, 3; \quad \beta^2 = \mathbb{I}; \quad \alpha^i \alpha^j + \alpha^j \alpha^i = 0, \quad i \neq j; \quad \alpha^i \beta + \beta \alpha^i = 0, \quad i \neq j; \]

  \[ \gamma^0 = \beta, \quad \gamma^i = \beta \alpha^i, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{I}, \quad \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \]

- Dirac representation:
  
  \[ \alpha^i = \sigma^1 \otimes \sigma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad \beta = \sigma^3 \otimes \mathbb{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \]

  \[ \gamma^0 = \beta, \quad \gamma^i = \beta \alpha^i = i\sigma^2 \otimes \sigma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \quad \gamma_5 = \sigma^1 \otimes \mathbb{I} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad \gamma_5^2 = \mathbb{I}. \]

where the Pauli matrices are

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Note that $\alpha^i$, $\beta$ and $\gamma^0$ are Hermitian, whereas the $\gamma^i$ are anti-Hermitian.

Note: Throughout the paper we use units where $\hbar = c = 1$. 

End of Paper