GREEN'S FUNCTIONS FOR THE SCALAR FIELD

Imagine we wish to solve the following INHOMOGENEOUS partial differential equation:

\[-(\Box + m^2) \phi(x) = J(x)\]

where \(J(x)\) is a "source term".

The associated HOMOGENEOUS EQUATION is of course

\[(\Box + m^2) \phi(x) = 0, \quad \text{the KG equation.}\]

As usual, the most general solution to the inhomogeneous equation can be written as

\[\phi(x) = \phi_p(x) + \int d^4x' G(x-x') J(x')\]

where \(\phi_p(x)\) is a particular solution of the homogeneous equation and \(G(x,x')\) is the Green function defined via

\[(\Box + m^2) G(x) = -\delta^{(4)}(x)\]

Indeed

\[-(\Box + m^2) \phi(x) = -(\Box + m^2) \phi_p +
- (\Box + m^2) \int d^4x' G(x-x') J(x') =
= 0 + \int d^4x' \delta^{(4)}(x-x') J(x') = J(x).\]
There are infinitely many Green's functions as I can always add a solution to the homogeneous equation and obtain another Green function.

**SOLVING THE GREEN FUNCTION EQUATION**

We wish to solve

\[ (D + m^2) G(x) = - \delta(x) \]

In order to do so, we go to Fourier space – our convention for Fourier transforms:

\[ G(x) = \frac{1}{(2\pi)^d} \int \hat{G}(k) e^{i(kx)} \]

\[ \hat{G}(k) = \int d^d x \ e^{i(kx)} \ G(x) \]

Using the first relation, we have

\[ - (D + m^2) G(x) = \frac{1}{(2\pi)^d} \int \frac{d^d k}{e^{\frac{-ikx}{(k^2 + m^2)^2}}} \hat{G}(k) = \]

\[ = \frac{1}{(2\pi)^d} \int \frac{d^d k}{e^{\frac{-ikx}{(k^2 + m^2)^2}}} \hat{G}(k) = \delta(x) \]

Now use \( \delta(x) = \frac{1}{(2\pi)^d} \int \frac{d^d k}{e^{\frac{-ikx}{(k^2 + m^2)^2}}} \)

and one would conclude that
\[ G(k) = \frac{1}{k^2 - m^2} \]

Then one would conclude

\[ G(x) = \int e^{-ikx} \frac{1}{k^2 - m^2} \]

However, there is a problem related to the \( k \) integration. Indeed, the denominator is

\[ k^2 - m^2 = k^2 - (k + m)(k - m) = k^2 - \omega(k) = (k_0 - \omega(k))(k_0 + \omega(k)) \]

hence the are 2 poles along the integration path of \( k_0 \) at \( k_0 = \pm \omega(k) \)

Hence \( G(x) \) written as before does not make sense. And we need to give prescriptions on how to go around poles.

As we shall see, different prescriptions will differ by solutions of the homogeneous equation as expected from our previous solution.
To study this problem, we focus on the \( k_0 \) integration and we regard \( k_0 \) as a complex variable: we complexify \( k_0 \).

We then consider deformations of the integration path. Integration paths that can be deformed into each other without crossing any of the singularities leave the integral invariant.

So we consider the following integral

\[
\int \frac{d k_0}{(2\pi)^{1}} \int_{C} \frac{d k}{(2\pi)^{1}} \frac{e^{-i k_0 x} e^{i k x}}{(k_0 - \omega(k))(k_0 + \omega(k))}
\]

where \( C = \text{curve in the complex } k_0 \text{ plane.} \)

I will now introduce 3 popular new integration paths and we will compute the Green function in these cases.

(A) \hspace{2cm} \text{RETARDED}

(B) \hspace{2cm} \text{ADVANCED}

(C) \hspace{2cm} \text{FEYNMAN}
NO-FRILLS COMPUTATION OF $\Delta_F(x)$

We define

$$\Delta_F(x) = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2}$$

First we rewrite $k^2 - m^2 = k_0^2 - \omega^2 = (k_0 - E(k))(k_0 + E(k))$.

Then look at contour: if I close above,

\[ \]  

This gives damping for $R \to \infty$ only when $t < 0$.

Because the contour is anti-clockwise, the result is

$$t < 0 : +2\pi i \quad \text{Res} \quad \int \frac{d^3 k}{(2\pi)^3} e^{-iRk^2}$$

$$= -i \int \frac{d^3 k}{(2\pi)^3} \frac{iE(k)k + iE(k)x}{2E(k)^2} = -i \int \frac{d^3 k}{(2\pi)^3} \frac{ikx}{2E(k)}$$

Changing $\vec{k} \to -\vec{k}$
For $t \gg \hbar/k$, we have:

Now the causal is clockwise oriented $\Rightarrow$ next is

$$t \gg \hbar/k: \quad -2\pi \text{ Res} \quad \sum_{k_0 = E(k)} \frac{d^3 k}{(2\pi)^4} \frac{e^{-i k \cdot x}}{(k_0 - E(k))(k_0 + E(k))}$$

$$= -i \int \frac{d^3 k}{(2\pi)^3} \frac{e^{-i k \cdot x} E(k)}{2E(k)}$$

$$= -i \int \frac{d^3 k}{(2\pi)^3} \frac{e^{-i k \cdot x}}{2E(k)}$$

Summary:

$$\Delta F(x) = -i \int \frac{d^3 k}{(2\pi)^3 (2E(k))} \left[ e^{-i k x} \theta(x_0) + e^{i k x} \theta(-x_0) \right]$$

This is conventionally written as

$$i \Delta F(x) = i \Delta^{(+)}(x) \theta(x_0) - i \Delta^{(-)}(x) \theta(-x_0)$$

with

$$i \Delta^{(+)}(x) = \pm \int \frac{d^3 k}{(2\pi)^3 (2E(k))} e^{-i k x}$$
THE COMMUTATOR $[\phi(x), \phi(0)] = i \Delta(x)$ AND CAUSALITY

For complex Klein-Gordon (for real: $[\phi(x), \phi(0)]$).

Using $\phi(x) = \int d^3 k \ N(k) \left[ a(k) e^{-ikx} + b(k) e^{ikx} \right]$

$\Phi(0) = \int d^3 q \ N(q) \left[ a(q) + b(q) \right]$

$[\phi(x), \Phi(0)] = \int d^3 k \ d^3 q \ N(k) N(q) \cdot$

$\left\{ \left[ a(k), a^+(q) \right] e^{-ikx} + \left[ b^+(k), b(q) \right] e^{ikx} \right\}$

Since $[a(k), a^+(q)] = [b^+(k), b(q)] = (2\pi)^3 (2\epsilon(k)) \delta^{(3)}(k-q)$

in the nonrelativistic where $N(k) = \frac{1}{(2\pi)^3 (2\epsilon(k))}$

we get

$[\phi(x), \Phi(0)] = \int \frac{d^3 k}{(2\pi)^3 2\epsilon(k)} \left[ e^{-ikx} + e^{ikx} \right]$

$= i \Delta^{(+)}(x) + i \Delta^{(-)}(x)$ with

$\Delta^{(\pm)}(x) = \pm \int \frac{d^3 k}{(2\pi)^3 2\epsilon(k)} e^{\mp ikx}$

It is customary to define
\[ \Delta(x) = \Delta^+(x) + \Delta^-(x) \]  

thus

\[ [\phi(x), \phi^+(0)] = \lambda \Delta(x). \]

\textbf{Comment:}

\( \Delta^+(x) \) are relativistic invariant.

Indeed the integration measure is

\[ \frac{d^3k}{2E(k)} \]

and

\[ \l_1, k \cdot x \]

are relativistic invariant.

\( \Delta^- \) are functions of \( x^2 \)

(he only relativistic invariant quantity around).

Now, if the fields are observables, we need that

\[ [\phi(x), \phi^+(0)] = 0 \text{ for } x = \text{ spacelike}. \]

We can then go to a frame where \( x = (0, \vec{x}) \).

That means we compute the equal time commutator.
This is \[ \phi(x,0) \phi^+(\vec{r},0) = i \delta(\vec{x},\vec{r}) = \]
\[
= \int \frac{d^3k}{(2\pi)^3 2\varepsilon(k)} \begin{bmatrix} e^{-iE\cdot0 + i\vec{k} \cdot \vec{r}} + iE\cdot0 - i\vec{k} \cdot \vec{r} \\ e + e \end{bmatrix} 
\]
\[
= \int \frac{d^3k}{(2\pi)^3 (2\varepsilon(k))} \begin{bmatrix} +i\vec{k} \cdot \vec{r} & -i\vec{k} \cdot \vec{r} \\ e - e \end{bmatrix} = 0,
\]

since we can perform charge integration variable, say, with first term from \[ \vec{k} \rightarrow -\vec{k} \], thus cancelling the second term.

Hence \[ \phi(x), \phi^+(\vec{r}) \] vanishes at spacelike distances.
A nice way to rewrite $\Delta_F(x)$

We defined the Feynman propagator as

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int \frac{d^4k}{k^2 - m^2} \ e^{-ikx}$$

We can move the poles while deforming the contour in such a way that the poles do not cross the contour without altering the result of the integral.

We do so in this way:

With the new contour being the real axis.

The new poles must be at $\mp (\omega(k) - i\varepsilon)$
In order to achieve so, our denominator
\[ \frac{2}{2} \frac{2}{2} \frac{2}{2} \frac{2}{2} \]
\[ K - m = \frac{K_0 - \omega(k)}{k_0 + \omega} \]

must be deformed into
\[ (K_0 - (\omega + i\epsilon))(K_0 - (-\omega + i\epsilon)) = \]
\[ = (K_0 - \omega + i\epsilon)(K_0 + \omega - i\epsilon) = \]
\[ = (K_0 - \omega)(K_0 + \omega) + i\epsilon(K_0 + \omega - K_0 + \omega) + \epsilon^2 \]
\[ = K^2 - m^2 + 2i\epsilon\omega \]

same as \( i\epsilon \) since \( \omega > 0 \)
and \( \epsilon \) is infinitesimal
\[ = K^2 - m^2 + i\epsilon \]
\[ \Rightarrow \]
\[ K^2 - m^2 \Rightarrow \text{deform to} \quad K^2 - m^2 + i\epsilon \]

Here we can write
\[ \Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{K^2 - m^2 + i\epsilon} \]

The Feynman propagator will play an important role in what follows.
The best way to feel gratified by the calculations we did earlier is to put them to use.

To this end, we introduce the following quantity:

\[ T(\phi(x)\phi^+(y)) = \Theta(x_0-y_0)\phi(x)\phi^+(y) + \Theta(y_0-x_0)\phi^+(y)\phi(x) \]

which we call \textit{time-ordered}, or \textit{T-product}.

We wish to show that:

1. Its vacuum expectation value \[ \langle 0 | T(\phi(x)\phi^+(y)) | 0 \rangle \] has a nice physical interpretation, describing

   - The creation of a particle at point \( y \) and its subsequent annihilation at \( x \) if \( x_0 > y_0 \), or

   - The creation of an antiparticle at point \( y \) and its subsequent annihilation if \( x_0 < y_0 \).

2. That quantity is nothing but \( \lambda \Delta_E(x-y) \)

   \( \text{i.e. the Feynman propagator} \)
Firstly we expand our T-product out:

\[ \Phi = \Phi^+ + \Phi^- \]

\[ \Phi^+ = (\Phi^+)^+(+) + (\Phi^+)^+(-) \]

\[ \Phi^- = (\Phi^-)^+(-) + (\Phi^-)^+(-) \]

Where \( \pm \) denotes positive/negative frequency.

From:

\[ \Phi \sim a e^+ + b e^- \]

\[ \Phi^+ \sim b^+ \]

\[ \Phi^- \sim b^- \]

We see that:

\[ \langle 0 | \Phi^+ | 0 \rangle = \langle 0 | \Phi^-(+) | 0 \rangle = 0 \]

\[ \langle 0 | \Phi^- | 0 \rangle = \langle 0 | \Phi^+(+) | 0 \rangle = 0 \]

Hence:

\[ \langle 0 | T(\Phi(x), \Phi(y)) | 0 \rangle = \]

\[ = \Theta(x_0-x_0) \langle 0 | \Phi(x) \Phi^+(y) | 0 \rangle + \Theta(y_0-x_0) \langle 0 | \Phi(y) \Phi(x) | 0 \rangle \]

\[ = \Theta(x_0-x_0) \langle 0 | \Phi^+(-) \Phi^+(-) | 0 \rangle + \Theta(y_0-x_0) \langle 0 | \Phi^+(+) \Phi^+(-) | 0 \rangle \]

Let's look at the 2 cases separately:

i) For \( x_0 > y_0 \), only the first term contributes.

In this contribution, \( \Phi^+(-) \Phi^+(-) \) \( \exists A \) creates a particle at \( y \)

\[ \Phi(x) \in A \] annihilates the particle at \( x \)
ii) For $x < y$, only the second term contributes. Here

$\phi(x) \in \mathcal{H}$ creates a particle at $x$

$(\phi^†)^+ (y) \in \mathcal{H}$ destroys the anti-particle at $y$.
The explicit calculation of $\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle$

As we saw before,

$$\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle = \Theta(x_0 - y_0) \langle 0 | \phi(x)\phi(y) | 0 \rangle +$$

$$+ \Theta(y_0 - x_0) \langle 0 | \phi(y)\phi(x) | 0 \rangle =$$

$$= \Theta(x_0 - y_0) \langle 0 |^{(1)} \phi(x) \phi(y) | 0 \rangle +$$

$$+ \Theta(y_0 - x_0) \langle 0 |^{(2)} \phi(y) \phi(x) | 0 \rangle$$

We compute separately the two vacuum expectation values:

1) $\langle 0 |^{(1)} \phi(x) \phi(y) | 0 \rangle =$

$$= \int \frac{d^3k}{(2\pi)^3} e^{-i(kx + ky)} \langle 0 | a(k) a^+(k') | 0 \rangle$$

where $\hat{a}(k) = \frac{\partial^3}{(2\pi)^3 (2\omega_k)}$ is the

relativistic invariant normalization.

We now observe that

$$\langle 0 | a(k) a^+(k') | 0 \rangle = \langle 0 | E a(k), a^+(k') | 0 \rangle$$

since the added term is equal to zero.

Next we use $[a(k), a^+(k')] = \frac{\partial^3}{(2\pi)^3 (2\omega_k)} \delta(k-k')$

valid in the relativistic invariant normalization.
Thus we get

\[ \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3k}{(2\pi)^3(2\omega_k)} e^{i(kx+y)} \]

\[ = \int \frac{d^3k}{(2\pi)^3(2\omega_k)} e^{-i\omega_k(x_0-y_0)} e^{ikr} \]

Now recall that

\[ i \Delta(x) = \pm \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\omega_k x}}{2\omega_k} \]

We also have

\[ \langle 0 | \phi(y) \phi(x) | 0 \rangle = \langle 0 | \phi(y) | 0 \rangle^* \langle 0 | \phi(x) | 0 \rangle \]

\[ = \int \frac{d^3k}{(2\pi)^3(2\omega_k)} e^{-i\omega_k x} \]

\[ = \int \frac{d^3k}{(2\pi)^3(2\omega_k)} e^{i\omega_k x} \]

\[ = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik(x-y)}}{2\omega_k} = -i \Delta(x-y) \]

\[ = -i \Delta(x-y) \left. \right| \text{here} \]

\[ \]
\[ \langle 0 | \phi(y) \phi(x) | 0 \rangle = -\imath \Delta(x-y) \]

Putting things together we obtain

\[ \langle 0 | \mathcal{T}(\phi(x) \phi^\dagger(y)) | 0 \rangle = \]

\[ = \imath \Delta^{(+)}(x-y) \Theta(x_0-y_0) - \imath \Delta^{(-)}(y-y_0) \Theta(y_0-x_0) \]

But we had defined earlier the Feynman Green function \( \Delta_F(x) \) for which we had found

\[ \Delta_F(x) = \Delta^{(+)}(x) \Theta(x_0) - \Delta^{(-)}(x) \Theta(-x_0) \]

Here we conclude that

\[ \langle 0 | \mathcal{T}(\phi(x) \phi^\dagger(y)) | 0 \rangle = \imath \Delta_F(x-y) \]