In quantum mechanics, for a system of many degrees of freedom \( \{ q_n \} \), we quantise by requiring

\[
[q_n, p_s] = i \delta_{ns}
\]

where \( n, s = 1, \ldots, \# \text{ of degrees of freedom} \).

Now we have

\[
\mathcal{L} = \int d^3 x \, L (\phi (x, t), (\partial_t \phi)(x, t))
\]

\( \phi \) is like \( r \); if we were to discretise.

Then

\[
d^3 x \to SV_n
\]

\( SV_n \) = volume of cell \( n \) and

\[
\mathcal{L} \to \sum_n SV_n \, L (q_n, \dot{q}_n; q_{\# d.o.f.})
\]

depends only on \( q_n \).
so that

\[ \frac{\Theta L}{\partial q_n} = \delta V_n \frac{\Theta L}{\partial q_n} \]

or

\[ q_n = \frac{\Theta L}{\partial q_n} = \delta V_n \pi \]

(recall \( \pi(x,t) = \frac{\Theta L}{\partial (\partial \phi)(x,t)} \))

Hence \([q_n, p_s] = i \delta_{ns}\) becomes

\[
\begin{align*}
q_n(t) & \to \phi(x_n, t) \\
p_n(t) & \to \delta V_n \pi(x, t)
\end{align*}
\]

\[
\left[ \phi(x_n, t), \delta V_n \pi(x, t) \right] = i \delta_{ns}
\]

or

\[
\Gamma \phi(x_n, t), \pi(x, t) \right] = \frac{i \delta_{ns}}{\delta V_n}
\]

Now take the continuum limit:

\[ \delta V_n \to \frac{1}{\delta x} \text{ infinitesimal; the RHS is such that} \]

\[ \sum_n \frac{\delta V_n (i \delta_{ns})}{\delta V_n} \to \int \frac{1}{\delta x} \]

\[ \text{What does this become?} \]
We know that \( \int d^3 x \, \delta^{(3)}(x - y) = 1 \)

since we have derived that we have to impose

\[
[\phi(x, t), \pi(y, t)] = i \delta^{(3)}(x - y)
\]

This is called an **equal time commutation relation** or **canonical commutation relation**.

\([q, \pi] = 0\)

Now we wish to answer the question: What does this imply for the \(a\) and \(b\)?

We will prove that \(a\) and \(b\) become **operators** by proving that they no longer commute. In particular, we will prove that

**Theorem:**

\[
[a(k), a^+(k')] = [b(k), b^+(k')] = 0
\]

\[
[a(k), a^+(k')] = [b(k), b^+(k')] = \frac{\delta^{(3)}(k - k')}{\mathcal{N}(k)}
\]
Let's prove this, but let's already get intrigued by this formula which looks like we are going to deal with harmonic oscillators.

Ok, with this in mind, first we prove an "inversion formula" which allows us to go from $\Phi$ to $a$.

**Inversion Formula:**

\[
\frac{a(k)}{N} = \frac{1}{(2\pi)^3 (2\omega k^3)} \int d^3 x \ e^{i(k \cdot x)} \Phi(x)
\]

\[
\frac{b(k)}{N} = \frac{1}{(2\pi)^3 (2\omega k^3)} \int d^3 x \ e^{i(k \cdot x)} \Phi^+(x)
\]

**Proof:** We have $\int d^3 x \ e^{i(k \cdot x)} \Phi(x) = \text{using the } \Phi \text{ expansion}

\[
(\omega) = \int d^3 k \ N(k) \begin{bmatrix} a(k) \int d^3 x \ e^{i(k \cdot x)} (\omega) e + b(k) \int d^3 x \ e^{i(k \cdot x)} (\omega) e^+ \end{bmatrix}
\]

To proceed we show that

\[
\int d^3 x \ e^{i(k \cdot x)} = (2\pi)^3 (2\omega k^3) \delta^{(3)} (k - k')
\]

Indeed, \[
\int d^3 x \ e^{i(k \cdot x)} = e^{i(-1)(k_0' - (-1)k_0)} \int d^3 x \ e^{i(k - k') \cdot x}
\]

\[
= e^{i(\omega(k) - \omega(k'))} \int d^3 x \ e^{i(k - k') \cdot x} = \frac{1}{(2\pi)^3 \delta^{3}(k - k')}
\]
\[ a(\mathbf{k}) = \frac{4}{(2\pi)^3 2\omega(k) N(k)} \int d^3 \mathbf{x} e^{i \mathbf{k} \cdot \mathbf{x}} (\mathbf{a}_{\mathbf{k}}) \phi(x) \]

We will call the prefactor
\[ g(k) = \frac{4}{(2\pi)^3 2\omega(k) N(k)} \]

When \( g = 1 \) with the popular choice
\[ N = \left[ (2\pi)^3 2\omega(k) \right]^{-1} \]
Note that

\[ \phi = e^{-i k x} e^{i k x} \]

and therefore we also have

\[ b(k) = \frac{1}{(2\pi)^3(2\pi k^2)^{1/2}} \int dx \ e^{-i \varepsilon_0 \varphi(x)} \]

without the need of performing a new calculation.

**Exercise:** Check that \( a = b = 0 \)

**Calculation of \([a(k), a(k')]\), \([a(k), a^\dagger(k')]\) etc.**

We have (using our notation)

\[ g(k) = \frac{(2\pi)^3(2\pi k^2)^{1/2}}{1} \]

\[ [a(k), a(k')] = g(k) g(k') \int dx \ dy \left[ e^{-i \varepsilon_0 \varphi(x)} e^{-i \varepsilon_0 \varphi(y)} \right] \]

where the subscript "ET" stands for Equal Time, a reminder that here \( x_0 = y_0 \). Next we recall that

\[ [\phi(x, t), \phi(y, t)] = 0 \quad \text{(some for } [\phi(x^+ t), \phi(x^+ t)] = 0 \text{)} \]

and the only nonvanishing counter term is

\[ [\phi(x^+ t), \pi^+ (y^+ t)] = \frac{i}{\hbar} \delta^{(3)}(x-y) \quad [\phi(x^- t), \pi^+ (y^+ t)] = -\frac{i}{\hbar} \delta^{(3)}(x-y) \]

In complex KS:

\[ \pi^+ (x^+ t) = \frac{\partial \varphi}{\partial (\varphi^+)} (x^+ t) = (\varphi^+ \varphi^+)(x^+ t) \]

\[ \pi^+ (x^- t) = \frac{\partial \varphi}{\partial (\varphi^+)} (x^- t) = (\varphi^+ \varphi^+)(x^- t) \]
We see immediately that \([ a, a ] = 0 \).

Next, we look at

\[
\langle a(k), a(k') \rangle = g(k) g(k') \int d^3 x d^3 y \left\{ e^{i k x} \theta(x) - i k y \right\} e^{-i k x} \theta(x), e^{i k y} \theta(y) \right\}
\]

and we have to select only terms with nonvanishing \(\theta\) factors. We get

\[
= g(k) g(k') \int d^3 y \left\{ e^{i k x} \theta(x) - i k y \right\} \left\{ e^{-i k x} \theta(x), e^{i k y} \theta(y) \right\}
\]

\[
= g(k) g(k') \int d^3 y \left\{ e^{i k x} \theta(x) - i k y \right\} \left\{ e^{-i k x} \theta(x), e^{i k y} \theta(y) \right\}
\]

\[
= g(k) g(k') \int d^3 y \left\{ e^{i k x} \theta(x) - i k y \right\} \left\{ e^{-i k x} \theta(x), e^{i k y} \theta(y) \right\}
\]

Now,

\[
\left\{ e^{i k x} \theta(x), e^{i k y} \theta(y) \right\} = -i \delta(x - y)
\]

\[
\left\{ e^{-i k x} \theta(x), e^{i k y} \theta(y) \right\} = +i \delta(x - y)
\]

\[
\left\{ e^{-i k x} \theta(x), e^{i k y} \theta(y) \right\} = -i \delta(x - y)
\]

\[
\left\{ e^{i k x} \theta(x), e^{-i k y} \theta(y) \right\} = +i \delta(x - y)
\]

hence we obtain...
\[ [a^\dagger(k), a(k')^\dagger] = g(k) g(k') \int d^3x \, d^3y \, \langle - \rangle \, \langle - \rangle \, (e^{ikx} e^{-ik'y}) \langle x \rangle \langle y \rangle \delta^{(3)}(x-y) = \]

\[ = (g(k) g(k')) \int d^3x \, e^{ikx} \rightarrow (e^{-ik'y}) \]

(Where \( x = y \) because of \( \delta^{(3)}(x-y) \) and \( x_0 = y_0 \) because of \( E^+ \)).

Now,

\[ \int d^3x \, e^{ikx} \rightarrow (e^{-ik'y}) = \frac{1}{(2\pi)^3(2\omega(k))} \delta^{(3)}(k-k') \]

as we have proved earlier. Hence

\[ [a^\dagger(k), a(k')^\dagger] = g^2(k) g^2(k') \frac{1}{(2\pi)^3(2\omega(k))} \delta^{(3)}(k-k') \]

\[ [a^\dagger(k), a(k')^\dagger] = g^2(k) \frac{1}{(2\pi)^3(2\omega(k))} \delta^{(3)}(k-k') \]

\[ \text{PMK: In the relativistic, Lorentz normalisation, we have} \]

\[ N(k) = \frac{1}{(2\pi)^3(2\omega(k))}, \quad \text{we have} \]

\[ g(k^2) = \frac{1}{(2\pi)^3(2\omega(k)) N(k)} = 1 \]
Hence in the relativistic invariant model set up
the commutation relations read

\[
[a(k), a^+(k')] = \left(\frac{2\pi}{2\omega(k)}\right)^3 \delta^{(3)}(k - k')
\]

\[
[b(k), b^+(k')] = \left(\frac{2\pi}{2\omega(k)}\right)^3 \delta^{(3)}(k - k')
\]
SUMMARISING:

By imposing canonical commutation relations on the fields and their derivatives, these become operators. Hence

\[ a(\mathbf{n}), b(\mathbf{n}) \text{ BECOME OPERATORS THEMSELVES.} \]

NOTE: Had we used the discrete expansion

\[ \phi(\mathbf{x}, t) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\pi v}} \left[ a(\mathbf{n}) e^{-ikx} + b(\mathbf{n})^* e^{ikx} \right] \]

The commutation relations would be

\[ [a(\mathbf{n}), a(\mathbf{n}')^+] = \varepsilon_{\mathbf{n}, \mathbf{n}'} \]
We compute directly \([ \phi(x,t), \pi_{\phi}(y,t) ] \) and impose that the RHS is \( i \delta(x-y) \).

The expansions are:

\[
\phi(x,t) = \int d^3k \ N(k^2) \left[ -a(k) e^{-i E(k)t + i k \cdot x} + b(k) e^{+i E(k)t - i k \cdot x} \right]
\]

\[
\pi_{\phi}(y,t) = (\partial_0 \phi)(y,t) = \int d^3q \ N(q^2) (i E(q)^2) \left[ a(q) e^{+i E(q)t - i q \cdot y} - b(q) e^{-i E(q)t + i q \cdot y} \right]
\]

\[
[ \phi(x,t), \pi_{\phi}(y,t) ] = \int d^3k \ d^3q \ N(k^2) N(q^2) (i E(q)) \cdot \left\{ [a(k), a^+(q)] e^{-i (E(k) - E(q)) t + i (k \cdot x - q \cdot y)} + [b(k), b^+(q)] e^{i (E(k) - E(q)) t - i (k \cdot x - q \cdot y)} \right. \\
- \left. [a(k), b^+(q)] e^{i (E(k) + E(q)) t + i (k \cdot x + q \cdot y)} + [b(k), a^+(q)] e^{-i (E(k) + E(q)) t + i (k \cdot x + q \cdot y)} \right\}
\]
Now we show that if we impose

hermitian oscillator commutator for \([a^+, a]\)

and \([b, b^+]\) we then get

\[
[\phi(\vec{x}, t), \pi_\phi(\vec{y}, \tau)] = i \delta(x-y)
\]

More precisely, let's impose

\[
[a(\vec{k}), a^+(\vec{q})] = [b(\vec{k}), b^+(\vec{q})] = c(\vec{k}) \delta(\vec{k} - \vec{q})
\]

\[
[a(\vec{k}), b(\vec{q})] = 0
\]

and we will also determine \(c(\vec{k})\) — Doing so,

we immediately obtain

\[
[\phi(\vec{x}, t), \pi_\phi(\vec{y}, \tau)] = \int d^3k \ N^2(k) \ (i E(k)) \cdot \mathcal{C}(\vec{k})
\]

\[
i \delta(\vec{x} - \vec{y})
\]

\[
\cdot \mathcal{E}(\vec{k})
\]

Comparing to \(\int d^3k \ e^{i \vec{k} \cdot (\vec{x} - \vec{y})} = (2\pi)^3 \delta(x-y)\), we see that we need to impose

\[
N^2(k) \ E(k) \ C^2(k) = \frac{1}{(2\pi)^3} \quad \text{or} \quad C(k) = \frac{1}{(2\pi)^3 (2E(k)) N^2(k)}
\]
For the popular choice of $N(k^2) = \frac{1}{(2\pi)^3 (2\epsilon k^2)}$ one would get $c(k^2) = (2\pi)^3 (2\epsilon k^2)$ or

$$\left[ a(k^2), a^+(q^2) \right] = \left[ b(k^2), b^+(q^2) \right] = (2\pi)^3 (2\epsilon k^2) S^{(3)}(k^2-q^2)$$
SPACE OF STATES

The Hilbert space acted upon by the \( a \) and \( b \) operators contains:

(A) THE VACUUM STATE \( |0\rangle \)

It is defined as the state annihilated by the \( a \)'s and the \( b \)'s:

\[
\begin{align*}
a(k) |0\rangle &= 0 \\
b(k) |0\rangle &= 0
\end{align*}
\]

(B) one- and multi-particle states:

\[
\left| \eta_{p_1}^{(k_1)} ... \eta_{p_p}^{(k_p)}; m_1^{(\bar{p}_1)} ... m_q^{(\bar{p}_q)} \right>
\]

\[
= \frac{1}{\sqrt{n_1! ... n_p! m_1! ... m_q!}}
\]

\[
\left[ a^\dagger(k_1) \right]^{m_1} ... \left[ a(k_p) \right]^{m_p} \left[ b^\dagger(\bar{p}_1) \right]^{m_1} ... \left[ b(\bar{p}_q) \right]^{m_q} |0\rangle
\]

(Recall that for a single \( h.c. \) oscillator, the properly normalised state is)

\[
\left| n \right> = \frac{(a^\dagger)^n}{\sqrt{n!}}
\]

To understand the physical meaning of these oscillators, we consider the physical observables, \( \mathbf{H} \) and \( \mathbf{P} \), and of the Noether charge \( Q \) associated to phase transformations.
Calculation of $H$

\[ H = \int d^3x \, \mathcal{L} \]

with

\[ \mathcal{L} = \Pi_\phi (\partial_\mu \phi) + \Pi_\phi^+ (\partial_\mu \phi^+) - \mathcal{L} \]

where

\[ \Pi_\phi = (\partial_+ \phi^+) \quad \Pi_\phi^+ = (\partial_+ \phi) \]

\[ \mathcal{L} = (\partial_\mu \phi)^+ (\partial_\mu \phi) + (\partial_\mu \phi) (\partial_\mu \phi^+) - (\partial_\mu \phi)^+ (\partial_\mu \phi) + m^2 \phi^+ \phi \]

At this point we haven't quantised yet — we are classical — hence I can rewrite

\[ (\partial_\mu \phi)^+ (\partial_\mu \phi) \rightarrow (\partial_\mu \phi)^+ (\partial_\mu \phi) \]

Then, use

\[ (\partial_\mu \phi)^+ (\partial_\mu \phi) = (\partial_\mu \phi)^+ (\partial_\mu \phi) - (\partial_\mu \phi)^+ (\partial_\mu \phi) \]

\[ \mathcal{L} = (\partial_\mu \phi)^+ (\partial_\mu \phi) + (\partial_\mu \phi)^+ (\partial_\mu \phi) + m^2 \phi^+ \phi \]

Next we evaluate this on the solution to the KG equation. As we will quantise we will not allow ourselves to move around the $\phi$ and $\phi^+$ equations. Starting point will be:

\[ \phi(\vec{r}^\prime) = \frac{1}{\sqrt{2\pi \hbar^2}} \int d^3p \, N(p) \left[ e^{-i p \cdot \vec{r}^\prime} + e^{i p \cdot \vec{r}^\prime} \right] \]

(with $p_0 = \omega(p)$ here) and

\[ \Gamma(a(p), a^+(p^\prime)) = (2\pi)^3 (2\omega(p^\prime)) \delta(p - p^\prime) \]

\[ x_f = N(p^\prime) = \frac{1}{(2\pi)^3 (2\omega(p^\prime))} \]
We have to compute \( \int d^3x \cdot \frac{\partial^2}{\partial x^2} x \cdot \frac{\partial^2}{\partial x^2} \cdot \text{W}\). Let's begin with the 1st term:

\[
\int d^3x \cdot \left( \frac{\partial^2}{\partial x^2} x \cdot \frac{\partial^2}{\partial x^2} \cdot \text{W} \right) = \int d^3x \int d^3p \cdot \text{W}^* \cdot N(p) \cdot N(p')
\]

\[
\int d^3p \cdot \text{W}^* \cdot \left[ e^{i\omega(p')} - e^{-i\omega(p')} \right]
\]

\[
\text{Next we use} \quad \int d^3x \cdot e^{-i(p+p')x} = \int d^3x \cdot e^{i\omega(p+p')x} = e^{i\omega(p+p')x} \int d^3x \cdot e^{-i\omega(p+p')x} = (2\pi)^3 \delta^{(3)}(p+p')
\]

\[
= \left\{ \begin{array}{l}
-2i\omega(p')t \\
(2\pi)^3 \delta^{(3)}(p+p')
\end{array} \right. \quad \text{to get}
\]

\[
\int d^3p d^3p' \cdot \delta^{(3)}(p-p') \cdot \omega(p) \omega(p') = (2\pi)^3
\]

\[
\left\{ \begin{array}{l}
[a(p) a(p') + b(p) b(p')] \delta^{(3)}(p-p') + \\
2i\omega(p)t \cdot \left[ a(p) b(p') + b(p) a(p') \right] \delta^{(3)}(p-p')
\end{array} \right.
\]
\[ \int d^3x \left( \hat{A} \phi \right)^T (\hat{A} \phi) = \int d^3p \int d^3p' N(p) N(p') \left[ a^+(p) a(p) + b(p) b^+(p') \right] (2\pi)^3. \]

\[ \left\{ a^+(p) a(p) + b(p) b^+(p) \right\} - \left[ a^+(p) b(-p) e^{2i\omega(p)t} + b(p) a(-p) e^{-2i\omega(p)t} \right] \]

Next we compute

\[ \int d^3x \left( \hat{\nabla} \phi \right)^T (\hat{\nabla} \phi) = \int d^3x \int d^3p \int d^3p' N(p) N(p') \left[ e^{i\hat{p} \cdot \hat{x}} a^+(p) - e^{-i\hat{p} \cdot \hat{x}} b(p) \right] . \]

\[ \left( -i \hat{p} \cdot \hat{x} \right) \left[ e^{i\hat{p} \cdot \hat{x}} a^+(p) - e^{-i\hat{p} \cdot \hat{x}} b(p) \right] = \left[ e^{i\hat{p} \cdot \hat{x}} a(p) - e^{-i\hat{p} \cdot \hat{x}} b(p) \right] = \]

Similarly to before, and cutting some steps

\[ \Rightarrow \int d^3x \left( \hat{\nabla} \phi \right)^T (\hat{\nabla} \phi) = \int d^3p \int d^3p' N(p) N(p') \]

\[ \left\{ \hat{p} \cdot \hat{p}' \right\} \left[ a^+(p) a(p') + b(p) b^+(p') \right] (2\pi)^3 \delta^3(p - p') + \]

\[ 2i\omega(p)t \]

\[ - \left( \hat{p} \cdot \hat{p}' \right) \left[ a^+(p) b(p') e^{-2i\omega(p)t} + b(p) a(p') e^{2i\omega(p)t} \right] \int (2\pi)^3 \delta^3(p + p') \]

\[ \delta^3(p + p') \]
\[ \int d^3 x \left( \nabla \phi \right)^+ \left( \nabla \phi \right) = \int d^3 p \ N^2(p^2) \ . \ (p^2)^2 \ (2\pi)^3 \ . \ \left\{ \begin{array}{l} \left[ a^+(p^2) a(p^2) + b(p^2) b^+(p^2) \right] + \\
 \left[ (-1)^2 a^+(p^2) b^+(1-p^2) e^{-i\omega(p^2)t} + b(p^2) a(-p^2) e^{-2i\omega(p^2)t} \right] \end{array} \right\} \]

Finally we compute

\[ m^2 \int d^3 x \ \phi^\dagger \phi = \int d^3 x \ \int d^3 p \ p^2 \ N(p^2) N(p^{'2}) \ . \]

\[ \left[ e^{ipx} a(p^2) + e^{-ipx} b(p^2) \right] \left[ e^{ipx} a(p^{'2}) + e^{-ipx} b(p^{'2}) \right] \]

\[ = \int d^3 p \ N(p^2) \ m^2 \ (2\pi)^3 \ . \]

\[ \left\{ \begin{array}{l} \left[ a^+(p^2) a(p^2) + b(p^2) b^+(p^2) \right] + \\
 \left[ a^+(p^2) b^+(1-p^2) e^{-i\omega(p^2)t} + b(p^2) a(-p^2) e^{-2i\omega(p^2)t} \right] \end{array} \right\} \]

Putting everything together we get:
\[ H = \int d^3 p \; N(\mathbf{p}) \left( \frac{2\pi}{\hbar} \right)^3 \cdot \left[ \frac{1}{2} \mathbf{p}^2 + m^2 \right] + \left[ a(\mathbf{p}) a^\dagger(\mathbf{p}) + b(\mathbf{p}) b^\dagger(\mathbf{p}) \right] \left[ \omega(\mathbf{p}) + \mathbf{p}^2 \right] \]

Using our normalization

\[ \frac{N^2(\mathbf{p})}{(2\pi)^3 (2\omega(\mathbf{p}))} = \frac{1}{(2\pi)^3 (2\omega(\mathbf{p}))} \]

\[ = \frac{\int d^3 p \; \left( \frac{2\pi}{\hbar} \right)^3 2 \omega^2(\mathbf{p}) \left[ a^\dagger(\mathbf{p}) a(\mathbf{p}) + b(\mathbf{p}) b^\dagger(\mathbf{p}) \right] \left[ a^\dagger(\mathbf{p}) a(\mathbf{p}) + b(\mathbf{p}) b^\dagger(\mathbf{p}) \right]}{(2\pi)^3 (2\omega(\mathbf{p}))} \]

\[ = \frac{\int d^3 p \; \omega(\mathbf{p}) \left[ a^\dagger(\mathbf{p}) a(\mathbf{p}) + b(\mathbf{p}) b^\dagger(\mathbf{p}) \right]}{(2\pi)^3 (2\omega(\mathbf{p}))} \]

Where we set \[ \frac{d^3 p}{(2\pi)^3 (2\omega(\mathbf{p}))} \] (Notation of Feynman, see page 114, formula (3.35))
One can perform the same calculation for \( p \) and for \( q \) with the result

\[
\begin{align*}
\hat{p}^2 &= \int d^3p \left[ a^+(p) a(p) + b(p) b^+(p) \right] \\
\hat{q}^2 &= \int d^3p \left[ a^+(p) a(p) - b(p) b^+(p) \right]
\end{align*}
\]

What we do next is to rewrite the last term:

We do this because \( a^+(k) a^-(k) \) is the number operator for momentum \( k \).

[Recall, for the harmonic oscillator,
\[ E_n = a^+ a \text{ with } a^+ a |n\rangle = n |n\rangle \]
]

and we wish to obtain the same for \( \hat{b} \).

Thus use \( [b(k), b^+(k')] = (2\pi)^3 (2\omega k)^2 \delta^{(3)}(k-k') \) to arrive

\[
\begin{align*}
b(k') b^+(k) &= b^+(k') b(k) + \\
&\quad \quad \quad \quad + (2\pi)^3 (2\omega k)^2 \delta^{(3)}(k-k') \\
&= E_{N, b}(k) + (2\pi)^3 (2\omega k)^2 \delta^{(3)}(k-k')
\end{align*}
\]

Hence
\[ H = \int d^3 p \ w(p) \left[ \epsilon^a_{\alpha}(k) + \epsilon^a_{\bar{\alpha}}(k) \right] + \text{infinite constant} \]

\[ \mathbf{p} = \int \frac{d^3 p}{(2\pi)^3} \hat{p} \left[ \epsilon^\alpha_{\alpha}(k) + \epsilon^\bar{\alpha}_{\bar{\alpha}}(k) \right] + \mathbf{c} \]

\[ G = \int d^3 p \left[ \epsilon^\alpha_{\alpha}(k) - \epsilon^\bar{\alpha}_{\bar{\alpha}}(k) \right] + \text{infinite constant} \]

The infinite constant is

\[ \int d^3 p \ w(p) \ \frac{(2\pi)^3}{(2\epsilon(p))^2} \delta(p-p') \]

and it cancels when \( p \rightarrow \mathbf{p} \).

**EX** Check this statement

The infinite constant is the (infinite) energy in the vacuum and is \text{OBSERVABLE}. We can only measure fluctuations about this.

To "renormalize away" we define all operators in a way that

\[ \text{ALL THE ANNIHILATION OPERATORS SIT ON THE RHS} \]

\[ \text{ALL THE CREATION OPERATORS SIT ON THE LHS} \]

\[ \text{since} \quad a |0\rangle = 0, \quad \langle 0 | a^\dagger = 0, \quad \text{etc.} \]
We denote this by $\mathfrak{H}$: on call it
"NORMAL ORDERING" so that we redefine
\[ \mathfrak{H} = \int 2i \mathcal{H}(x) : \left[ \hat{\sigma}_3 \right] : \mathcal{H}^* \]
and redo

$\mathfrak{H} |0\rangle = \emptyset$

Remark: there is obviously an ambiguity in translating classical into quantum quantities since at the classical level, expressions such as $\phi \phi^*$ or $\phi^* \phi$ are equivalent.

Normal ordering is a clever way to address this ambiguity.

**PARTICLE INTERPRETATION**

Let's take a state such as
\[ a^+(\mathbf{k}) |0\rangle \]

What are the energy/momentum/charge of such a state? We see vividly that
\[ H \alpha(k) |0\rangle = \omega(k) \alpha(k) |0\rangle \]
\[ \hat{p} \alpha(k) |0\rangle = \hat{p} \alpha(k) |0\rangle \]
\[ Q \alpha(k) |0\rangle = + \alpha(k) |0\rangle \]

(Dis) similarly

\[ H \beta(k) |0\rangle = \omega(k) \beta(k) |0\rangle \]
\[ \hat{p} \beta(k) |0\rangle = \hat{p} \beta(k) |0\rangle \]
\[ Q \beta(k) |0\rangle = - \beta(k) |0\rangle \]

Here we can interpret:

A) \( \alpha(k) |0\rangle \) as the state of a relativistic particle of mass \( \frac{\hbar}{k} \) every \( \omega(k) = \sqrt{\beta^2 + m^2} \)

(hence mass \( m \)) and value of the charge \( Q \) equal to \( +1 \)

B) \( \beta(k) |0\rangle \) as the state of a relativistic particle of momentum \( \frac{\hbar}{k} \) energy \( \omega(k) = \sqrt{\beta^2 + m^2} \)

(hence mass \( m \)) and value of the charge \( Q \) equal to \( -1 \)
COMMENT ON MANY PARTICLES and FREE THEORY

If we had a multi-particle state, the energy of it would be just the sum of the energies of the separate particles. This shows that the theory we are quantising is a theory WITHOUT INTERACTIONS i.e. it is the FREE THEORY!
We have seen that the RHS is convention dependent. For instance, with relativistic invariant normalisation we have

\[ \left[ a(k), a^+(k') \right] = (2\pi)^3 \left( 2\omega_k \right) \delta^{(3)}(k-k') \]

I just want to point out that this is related to the choice of \( \langle \hat{\mathbf{r}} | \hat{\mathbf{r}}' \rangle \). Indeed,

\[ \langle 0 | [a(k), a^+(k')] | 0 \rangle = \langle 0 | a(k) a^+(k') | 0 \rangle = \]

\[ = \langle \hat{\mathbf{r}} | \hat{\mathbf{r}}' \rangle \]  

So for instance

\[ \langle \hat{\mathbf{r}} | \hat{\mathbf{r}}' \rangle = (2\pi)^3 \left( 2\omega_k \right) \delta^{(3)}(k-k') \]

with the relativistic invariant normalisation.