MTH700 – Research Methods in Mathematical Sciences
Lecture 1: Mathematical Research

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The systematic investigation into and study of materials and sources in order to establish facts and reach new conclusions. – Oxford English Dictionary
Research

The **systematic** investigation into and study of **materials and sources** in order to establish **facts** and reach new conclusions. – Oxford English Dictionary

If I have seen further it is by standing on the shoulders of **giants**. – Isaac Newton

**Fact**: consistent with objective reality or can be proven

**Scientific method**: falsifiable from evidence

**Mathematical proofs**: exhaustive reasoning, always true

- composed of “obvious” steps (inference rules)
- in theory: from “obvious” statements (axioms)
- in practice: from existing statements (theorems)
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- Sequential discovery, by building on previous discoveries
  
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- Mathematical proofs: exhaustive reasoning, always true
  
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  - in practice: from existing statements (theorems)
Ingredients

- Good understanding of “the literature”
  - previous discoveries
  - facts, as well as techniques used to get there
- Reasonable amount of intelligence and creativity
- Large amounts of hard work, persistence, discipline
  
  Inspiration exists, but it has to find you working.
  
  – Pablo Picasso
- Get ready to work on difficult, unsolved problems
- Get ready to be stuck
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- Research skills are useful also in “real-world” problem solving
  - identify existing approaches
  - develop a new approach
  - argue that approach is suitable
Understanding the Literature

- Read and learn
  - results and techniques
  - basic: lecture notes, textbooks
  - advanced: textbooks, surveys, research papers

- Understand, apply, and synthesize
  - understand and reproduce theories
  - apply techniques to reproduce known results
  - synthesize into a big picture

- Role of advisor: mostly limited to feedback
  - guidance on relevant literature
  - help when you get stuck

- Your role: independent work
  - read critically
  - record progress and organize knowledge
The Research Process

1. Understand literature, identify open problem
2. Conjecture: educated guess at solution
3. Confirm or refute conjecture
4. Return to 2 until problem is solved
5. Return to 1 and refine

Role of advisor

- Identify open problems, while you learn to do this
- Help when you are stuck

Your responsibility

- Everything else
- Initiative, discipline, creativity, hard work

Periodic meetings to track progress

- Come prepared with concrete questions
- Show you have tried everything, are really stuck
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Communication in Research

- Present progress to advisor or collaborators
- Write a report on problem and solution
- Present results at seminar or conference
- Publish results in journal

Goals
- organize knowledge, keep track of progress
- share your work
- receive feedback

In collaborative work feedback may be constant

Role of advisor: feedback
Your responsibility: everything else
The Importance of Being Stuck

- Be ready to get stuck
- Comes with problem being important and unsolved
- Research happens while you are stuck
- Why does the current approach not work?
- Could the current conjecture be false?

- Be persistent: learn what you can about problem and technique
- Be creative: look at problem from as many angles as possible

- Are you asking the right question?
- Is there a simpler variant of the problem?
- Can what you have learned be applied to other problems?
- Are you using the right techniques?
Science and Mathematics

Science, any system of knowledge that is concerned with the physical world and its phenomena and that entails unbiased observations and systematic experimentation. In general, a science involves a pursuit of knowledge covering general truths or the operations of fundamental laws. — Encyclopedia Britannica

- Social sciences: study of individuals and society
  - economics, sociology, psychology

- Natural sciences: study of nature
  - biology, chemistry, physics

- Formal sciences: study of abstract concepts
  - mathematics, logic, theoretical computer science
  - do not rely on empirical evidence
  - essential role as the language of other sciences

- Further distinction between pure and applied sciences
The Scientific Method

- The basis of natural sciences at least since the 17th century
- Introduces new laws by the following process
  1. conjecture: (educatedly) guess new theory
  2. predict: derive logical consequences
  3. experiment: test whether real world behaves as predicted
- If experiment disagrees with prediction, conjecture is false

Conjectures guided by law of parsimony (Occam’s razor)

Crucial property: falsifiability, rigorous attempt at refutation

Example: classical mechanics and Mercury’s orbit around Sun

Natural and social sciences
- conjecture can never be proven correct
- if we follow scientific method it can be falsified

Formal sciences, including mathematics
- conjecture can be proven correct or incorrect

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Mathematical Research
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Pure and Applied Mathematics

- Applied mathematics: mathematics as a tool
  - developed to meet specific needs, from 3000 BC
  - navigation, astronomy, physics, engineering, economics, . . .
  - central elements: models and theorems

- Pure mathematics: mathematics as object of study
  - study of abstract concepts, from 19th century onwards
  - not-necessarily-applied mathematics
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- Distinction is philosophical, boundaries not very clear
- Pure mathematics has applications
  - differential geometry in general relativity
  - number theory in cryptography
- Applications lead to new areas of pure mathematics
  - differential equations
Pure and Applied Mathematics in Research

- Most mathematicians tend toward one of the two approaches
- In research, a combination of both is useful
- Therefore useful to have training in both

- Typical examples of a pure approach
  - property is interesting if it leads to interesting results
  - maximum level of generality at which a statement holds
  - minimum requirements for a statement to hold

- Typical examples of an applied approach
  - property is interesting if it describes a real-world phenomenon
  - application of different techniques to a problem
  - application of a technique to different problems

- Example of a combined approach
  - model a real-world phenomenon
  - look at simplest unsolved special case
Important aspect is quality, not purity

Quality is almost entirely unrelated to purity

https://xkcd.com/435/
Good Mathematics

- Terence Tao gives the following examples, and more

- Useful to keep these in mind as you do research

- Problem solving: major breakthrough on important problem

- Theory: concepts that simplify and generalize existing results

- Discovery: phenomenon, connection, counterexample

- Application: to the sciences or another area of mathematics

- Rigorous: all details correctly and carefully given in full

- Beautiful: easy and pretty to state but not to prove

- Elegant: achieving a difficult result with minimum effort

- Creative: radically new technique, viewpoint, type of result

- Strong: sharp result, unexpectedly strong conclusion

- Intuitive: natural and easily visualizable
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Timothy Gowers comments on two cultures in mathematics

W. T. Gowers. The Two Cultures of Mathematics.

Useful to be aware of them, benefit from both

Consider the following statements
1. point of solving problems is to understand mathematics
2. point of understanding mathematics is to solve problems

Many mathematicians agree with both, but not equally strongly
Those who agree more with the first: problem solvers
Those who agree more with the second: theory builders

Prime examples: Paul Erdős and Michael Atiyah

Emphasis dictated to certain extent by area
Examples: graph theory and algebraic geometry
Summary

- Scientific method: falsifiable from evidence
- Mathematical theorem: exhaustive reasoning, always true
- Occam’s razor: simpler is more plausible
- Research: understanding of literature and hard work
- Main activities: reading, writing, presenting
- Get comfortable with being stuck
- Role of advisor: initial direction, then mostly limited to feedback
- There are many measures of quality in mathematics
- You want to measure your own work against these
- Problem solving vs. theory building, pure vs. applied
- You want to be somewhere in between