POLYNOMIALS.

Introduction.

An algebraic expression $P(x)$ of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \qquad (1)$$

where $n \in \mathrm{N}$ i.e. where n is a positive integer or zero, and where $a_n, a_{n-1}, \ldots, a_1, a_0$ are given constants, $a_n \neq 0$, is called *a polynomial of degree n in x.*

We refer to the constants $a_n, a_{n-1}, \ldots, a_1, a_0$ as being the *coefficients of the powers of x*. Thus a_r is the coefficient of x^r for any $r \in \mathrm{N}$.

We see that by definition, corresponding to $n = 0$ we have the polynomial:

$$P(x) = a_0 \qquad (2)$$

We thus regard any constant as being a polynomial of degree zero.

If $n = 1$ i.e. if $P(x)$ takes the form:

$$P(x) = a_1 x + a_0 \qquad (3)$$

then we refer to $P(x)$ as being *a linear polynomial in x.*

Likewise, if $P(x)$ is of degree two, we say that it is *a quadratic polynomial in x*; if of degree three, *a cubic polynomial in x*, and so on.

Given that α is a particular value of x, we take $P(\alpha)$ to mean the value of $P(x)$ when $x = \alpha$ i.e.

$$P(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0 \qquad (4)$$

Thus for example, given that:

$$P(x) = x^3 - 2x^2 + x - 1$$

then:

$$P(-1) = (-1)^3 - 2(-1)^2 + (-1) - 1 = -1 - 2 - 1 - 1 = -5$$

If in particular:

$$P(\alpha) = 0 \qquad (5)$$

then we say that α *is a zero of the polynomial $P(x)$, or that it is a root of the equation* $P(x) = 0$.

We note that α may be a real, or complex number i.e. it is assumed that *x may take on real, or complex values.*

As to *the coefficients*, they may also be real or complex numbers. However, for now we shall assume them to be only *real* numbers, and talk in terms of *a polynomial with real coefficients.*

The Division Of One Polynomial By Another.

We consider the division of a polynomial $P(x)$ of degree m in x by another polynomial $Q(x)$ of degree n in x, where $P(x)$ and $Q(x)$ have real coefficients and $m \geq n$.

Through the normal steps of *long division*, carried out to the stage at which the first term of the *divisor* $Q(x)$ is of greater degree than that of the first term of the remaining expression of the *dividend* $P(x)$, it can be shown that there exists unique polynomials $S(x)$ and $R(x)$, with real coefficients, such that:

$$P(x) \equiv S(x)Q(x) + R(x) \tag{1}$$

where in particular $S(x)$, *the quotient* of the division, is always of degree $(m - n)$ in x, and $R(x)$, *the remainder* of the division, is always a polynomial of degree *at most*, $(n - 1)$ in x.

We give the result without going into the proof, which is somewhat algebraically 'heavy', involving as it does, the process of long division in its most general form. However, we illustrate the facts by the following example.

Let $P(x)$ and $Q(x)$ be given by:

$$\begin{aligned} P(x) &= x^5 + x^4 + 3x^3 + x + 1 \\ Q(x) &= x^2 + x + 1 \end{aligned} \tag{2}$$

Then using the process of long division, also known as *the division algorithm*, we have on dividing $P(x)$ by $Q(x)$, the following:

$$\begin{array}{llll} \text{DIVISOR:} & \text{DIVIDEND:} & & \text{QUOTIENT:} \\ x^2+x+1 \;) & x^5+x^4+3x^3+x+1 & (& x^3+2x-2 \\ & \underline{x^5+x^4+x^3} & & \\ & \quad 2x^3+x+1 & & \\ & \quad \underline{2x^3+2x^2+2x} & & \\ & \qquad -2x^2-x+1 & & \\ & \qquad -2x^2-2x-2 & & \\ & \text{REMAINDER:} \qquad x+3 & & \end{array}$$

From the basic statement in connection with long division in Elementary Algebra, the above implies that:

$$x^5 + x^4 + 3x^3 + x + 1 \equiv (x^3 + 2x - 2)(x^2 + x + 1) + (x + 3) \tag{3}$$

We see from this that the degree of the quotient is indeed the difference in the degrees of the dividend and divisor i.e. $5 - 3 = 2$, and that the degree of the remainder is, as stated in the general case, less than that of the divisor, it being 1 in the example.

$$\frac{P(x)}{Q(x)} \equiv S(x) + \frac{R(x)}{Q(x)}, \qquad Q(x) \neq 0 \tag{4}$$

This is also an identity, in that the values of x for which either side hold, are the same. But only in (1) may we apply those values of x for which $Q(x) = 0$.

The Remainder And Factor Theorems.

Let $P(x)$ be a polynomial with real coefficients, and be of degree m in x, then *the remainder theorem* states that the remainder when $P(x)$ is divided by $(x - \alpha)$, α a real constant, is $P(\alpha)$.

Now since $(x - \alpha)$ is of the first degree in x, then the remainder $R(x)$ on division of $P(x)$ by $(x - \alpha)$, must be of degree no greater than $(1 - 1)$ i.e. it must be of degree *zero*, and hence is independent of x. Denoting it by R, we may thus write that

$$P(x) \equiv S(x)(x - \alpha) + R$$

Putting $x = \alpha$, we then immediately have that

$$P(\alpha) = R \qquad (1)$$

as required to be shown.

The factor theorem, which in effect is a corollary of the remainder theorem, states that $(x - \alpha)$ is a factor of $P(x)$ if and only if

$$P(\alpha) = 0 \qquad (2)$$

Clearly if $P(\alpha) = 0$, then by the remainder theorem $R = 0$, and thus by definition of a factor, $(x - \alpha)$ is a factor of $P(x)$. Conversely, if $(x - \alpha)$ is a factor of $P(x)$, then R must be zero, and hence again by the remainder theorem, $P(\alpha)$ must also be zero.

Since $(x - \alpha)$ is linear, with real coefficients, we refer to it as being *a real linear factor* of $P(x)$ when $P(\alpha) = 0$. We note that here α is what we have termed *a zero of the polynomial* $P(x)$ i.e. it is *a real root of the polynomial equation* $P(x) = 0$.

Extension Of The Theorems.

Consider first the division of a polynomial $P(x)$ with real coefficients and of degree m in x by $(px + q)$, where p and q are real constants.

The degree of $(px + q)$ being one, the remainder, as before, will be independent of x, and thus we may write that

$$P(x) \equiv S(x)(px + q) + R \qquad (1)$$

where $S(x)$ is the quotient of the division, and R the remainder is constant.

Putting $x = -{}^{q}/_{p}$, we immediately have that

$$P(-{}^{q}/_{p}) = R \qquad (2)$$

We also have as a consequence of this that $(px + q)$ is a factor of $P(x)$ if and only if

$$P(-{}^{q}/_{p}) = 0 \qquad (3)$$

We note that on taking $p = 1$ and $q = -\alpha$, in the general linear form $(px + q)$, we have from (1) and (2) the results for the specific linear form $(x - \alpha)$.

Now some confusion may arise in connection with the extended theorem, in that by the basic theorem, statement (2) also implies that $P(-{}^{q}/_{p})$ is the remainder when $P(x)$ is divided by $(x + {}^{q}/_{p})$. However, this is indeed true, for we may write (1) as

$$P(x) \equiv S(x)p(x + {}^{q}/_{p}) + R \qquad (4)$$

and hence $P(-{}^{q}/_{p})$ *is* also the remainder when dividing by $(x + {}^{q}/_{p})$.

In the same way, if $P(-{}^{q}/_{p}) = 0$, then we may either say that $(px + q)$ is a factor of $P(x)$, or alternatively that $(x + {}^{q}/_{p})$ is a factor. Here we have that

$$P(x) \equiv S(x)(px + q) \equiv S(x)p(x + {}^{q}/_{p}) \qquad (5)$$

Non Linear Divisors (Quadratic or more)

Now suppose that we divide the polynomial $P(x)$ of degree m by a polynomial with real coefficients and of degree n in x, $m \geq n$. Then in general we have that

$$P(x) \equiv S(x)Q(x) + R(x) \qquad (14)$$

where $S(x)$ is of degree $(m - n)$, and $R(x)$ is of degree no greater than $(n - 1)$, and both $S(x)$ and $R(x)$ have real coefficients.

The approach to finding the remainder $R(x)$ here, is to write it in the most general form of a polynomial of degree $(n - 1)$ in x, and then make use of the identity (14) to determine the unknown coefficients of $R(x)$.

For example if $Q(x)$ is a *quadratic polynomial*, then we know that at most $R(x)$ is of degree $(2 - 1)$ i.e. 1, and in consequence take

$$R(x) = ax + b \qquad (15)$$

i.e. we write that

$$P(x) \equiv S(x)Q(x) + (ax + b) \qquad (16)$$

and from this determine the unknowns a and b. We note that if these are both zero, then $Q(x)$ is a quadratic factor of $P(x)$.

When possible, it is better to factorise $Q(x)$ into real linear factors and thus for instance when $Q(x)$ is a quadratic polynomial, write that:

$$P(x) \equiv S(x)(px + q)(rx + s) + (ax + b) \qquad (17)$$

In this form it is immediately clear that a and b can be found by taking $x = -{}^{q}/_{p}$ and $x = -{}^{s}/_{r}$, both values eliminating $S(x)$ from the equations for a and b.

The exact same form (17) is of course used when considering the division of $P(x)$ by *the product of two linear polynomials* $(px + q)$ and $(rx + s)$.

As a particular example, consider the division of

$$P(x) = x^5 - 3x^3 + 2x^2 - x + 1 \tag{18}$$

by the quadratic polynomial $(x^2 - 1)$ i.e. by the product of the linear factors $(x + 1)$ and $(x - 1)$.

Here we have that we may write that:

$$P(x) \equiv S(x)(x + 1)(x - 1) + (ax + b)$$

From which on taking $x = +1$ and $x = -1$, we get:

$$P(+1) = a + b$$

$$P(-1) = -a + b$$

But by (17)

$$P(+1) = 1 - 3 + 2 - 1 + 1 = 0$$

$$P(-1) = -1 + 3 + 2 + 1 + 1 = 6$$

Hence we have that

$$a + b = 0$$

$$-a + b = 6$$

giving $a = -3$, $b = 3$.

It therefore follows that:

$$R(x) = -3x + 3 \tag{19}$$

is the remainder of the division.

Identical Polynomials.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$
$$Q(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0 \qquad (13)$$

are two polynomials of the same degree and such that

$$P(x) \equiv Q(x) \qquad (14)$$

then

$$P(x) - Q(x) = (a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \ldots + (a_1 - b_1)x + (a_0 - b_0)$$

is a polynomial of degree n which is identically equal to zero, and hence by the corollary, its coefficients are all zero i.e.

$$a_n - b_n = 0,\ a_{n-1} - b_{n-1} = 0,\ \ldots,\ a_1 - b_1 = 0,\ a_0 - b_0 = 0$$

giving

$$a_n = b_n,\ a_{n-1} = b_{n-1},\ \ldots,\ a_1 = b_1,\ a_0 = b_0 \qquad (15)$$

In other words, when the polynomials are identical, the coefficients of corresponding powers are equal, as required to show.

Approach to Factorisation

As a basic example of the approach to factorisation, consider the *cubic polynomial*

$$P(x) = x^3 + 2x^2 - x - 2 \qquad (1)$$

As we said earlier, such a polynomial either has one or three linear factors.

If it has the latter, then clearly they must take the form $(x + p)$, $(x + q)$ and $(x + r)$, for the coefficient of x^3 is unity.

We also see that since the constant term is -2, then p, q and r can only involve the numbers 1, -1, 2 or -2.

With these basic ideas in mind it seems reasonable that $(x + 1)$ could be a factor.

Checking for this we see that since

$$P(-1) = (-1)^3 + 2\times(-1)^2 - 1 - 2 = 0$$

then by the factor theorem, $(x + 1)$ is indeed a factor.

Again it seems worthwhile testing for $(x - 1)$, and in this respect we have that since

$$P(1) = 1^3 + 2\times1^2 - 1 - 2 = 0$$

then it too by the factor theorem, is a factor of $P(x)$.

Now the constant term in the product of $(x + 1)$ and $(x - 1)$ is -1, and hence we must have that the remaining linear factor is $(x + 2)$. As a check on this we have that

$$P(-2) = (-2)^3 + 2\times(-2)^2 - (-2) - 2 = 0$$

and hence this is so.

We therefore have that we may write (1) as

$$P(x) = x^3 + 2x^2 - x - 2 = (x + 1)(x - 1)(x + 2) \qquad (2)$$

It of course immediately follows from this that the roots of the equation:

$$P(x) = 0$$

are

$$x = -1,\ x = +1,\ x = -2 \qquad (3)$$