THE QUADRATIC FUNCTION.

Introduction.

A fundamental form of relation that arises between two variables x and y is that given by:

$$y = ax^2 + bx + c \qquad (1)$$

where a, b and c are real constants with $a \neq 0$.

We see by inspection that the relation is such that to each real value of the independent variable x in R, there corresponds only one finite real value of the dependent variable y.

Hence the relation defines *a function* $f(x)$ and is such as to have as *domain the set of all real numbers* R.

We refer to the function as being *the quadratic function*, and in general specify it by writing that it is the function:

$$f(x) = ax^2 + bx + c,\ x \in \mathrm{R} \qquad (2)$$

The curve of the function is called *the quadratic curve*.

Of significance in connection with the function is the quadratic equation

$$ax^2 + bx + c = 0 \qquad (3)$$

which arises when $y = 0$ i.e. when $f(x) = 0$.

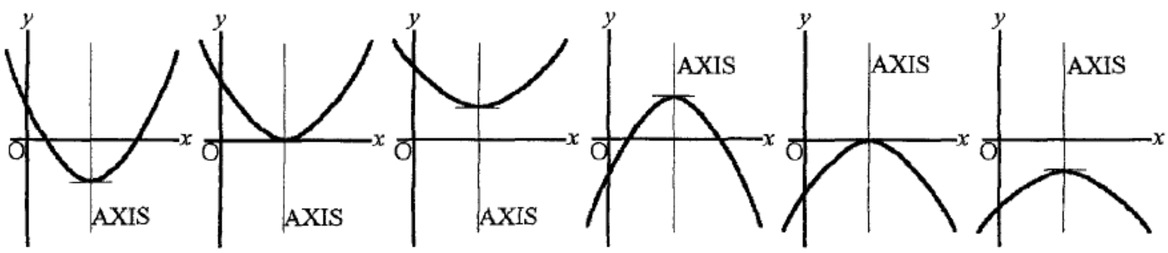
This we refer to as being *the associated quadratic equation*. Its importance in the analysis of the function will become apparent as we proceed, as will the significance of *the sign of a*.

The Six Fundamental Forms Of Quadratic Curves.

Given particular values of a, b and c and through tabulation plotting the graphs of the quadratic function

$$y = ax^2 + bx + c \qquad (1)$$

it will be found that the curve that is obtained fits into one of the following *six basic forms*:



PARABOLA THAT IS CONCAVE UPWARDS. PARABOLA THAT IS CONCAVE DOWNWARDS.

We now find this set, as a first step in the analysis of the quadratic function:

$$f(x) = ax^2 + bx + c, \;\; x \in \mathrm{R} \tag{1}$$

Applying the *technique for completing the square* to the right hand side of (1) we have that:

$$\begin{aligned} f(x) &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left\{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right\} \\ &= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right\} \end{aligned}$$

giving

$$f(x) = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

which on multiplying throughout by a gives:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \tag{2}$$

Now it is clear that for all $x \in \mathrm{R}$,

$$a\left(x + \frac{b}{2a}\right)^2 \geq 0, \quad a > 0 \tag{3}$$

and

$$a\left(x + \frac{b}{2a}\right)^2 \leq 0, \quad a < 0 \tag{4}$$

We also have that for a given a, b and c,

$$\frac{4ac - b^2}{4a} = K \tag{5}$$

a constant.

Thus *when a is positive*, we have by (2), (3) and (5) that the values of $f(x)$ as x takes on values in R, are determined on the basis of adding to a constant K, values that are greater than or equal to zero. The conclusion from this is that $f(x) \geq K$ for all $x \in \mathrm{R}$ i.e.

$$f(x) \geq \frac{4ac - b^2}{4a}, \quad a > 0 \tag{6}$$

for all values of $x \in \mathrm{R}$.

When a is negative, the values of $f(x)$ as x takes on values in R, are the result of adding to the constant values that are less than or equal to zero. The conclusion here is that $f(x) \leq K$ for all $x \in$ R i.e.

$$f(x) \leq \frac{4ac - b^2}{4a}, \quad a < 0 \tag{7}$$

for all values of $x \in$ R.

By results (6) and (7) we see that it can be stated that *when a is positive, $f(x)$ has a minimum value, and when a is negative it has a maximum value.*

Clearly, both these values occur when

$$x + \frac{b}{2a} = 0$$

i.e. when

$$x = -\frac{b}{2a} \tag{8}$$

and are such that

$$f_{\min} = f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}, \quad a > 0 \tag{9}$$

and

$$f_{\max} = f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}, \quad a < 0 \tag{10}$$

It follows from the above that *on accepting that the curve is in general parabolic with axis vertical, the curve of the function must be concave upwards when a is positive and concave downwards when a is negative.*

Now *when a is positive,* it is clear by (2), (3) and (5) that *as $x \to \pm\infty$, so $f(x) \to +\infty$.*

On the other hand, *when a is negative,* we see by (2), (4) and (5) that *as $x \to \pm\infty$, so $f(x) \to -\infty$.*

Putting the facts of the analysis together, we have that *over the domain* R, *the range or image set of $f(x)$ whenever a is positive is in terms of an interval given by*:

$$\frac{4ac - b^2}{4a} \leq f(x) < +\infty, \quad a > 0 \tag{11}$$

and whenever a is negative, it is specified by the interval:

$$-\infty < f(x) \leq \frac{4ac - b^2}{4a}, \quad a < 0 \tag{12}$$

By way of an example we determine on the basis of quoting the results, the image set of the function:

$$f(x) = x^2 + 3x - 1 \tag{13}$$

We immediately have that since $a = 1$ i.e. since a is positive, then $f(x)$ has a minimum value:

$$f_{\min} = \frac{4ac - b^2}{4a} = \frac{4 \times 1 \times (-1) - 3^2}{4 \times 1} = \frac{-4-9}{4} = -\frac{13}{4}$$

It follows that the image set of $f(x)$ is:

$$-\frac{13}{4} \le f(x) < +\infty \tag{14}$$

We note that again by the theory, the curve of the given example is concave upwards, and that the minimum value of the function occurs at

$$x = -\frac{b}{2a} = -\frac{3}{2} \tag{15}$$

It is left as an exercise to make a rough sketch of the curve on which should be clearly indicated the point at which the vertex of the curve occurs, and the values of $f(x)$ that form the image set.

We may of course approach the finding of the image set for a given function on *a first principles basis*. For the above example we have *on completing the square* that

$$\begin{aligned} f(x) &= x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 1 \\ &= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - 1 \end{aligned}$$

i.e.

$$f(x) = \left(x + \frac{3}{2}\right)^2 - \frac{13}{4} \tag{16}$$

From this we see that since for all $x \in \mathrm{R}$

$$\left(x + \frac{3}{2}\right)^2 \ge 0$$

then

$$f(x) \ge -\frac{13}{4}, \quad x \in \mathrm{R}$$

We also see that as $x \to \pm\infty$ so $f(x) \to +\infty$.

We therefore have that the image set is:

$$-\frac{13}{4} \le f(x) < +\infty$$

as previously determined.

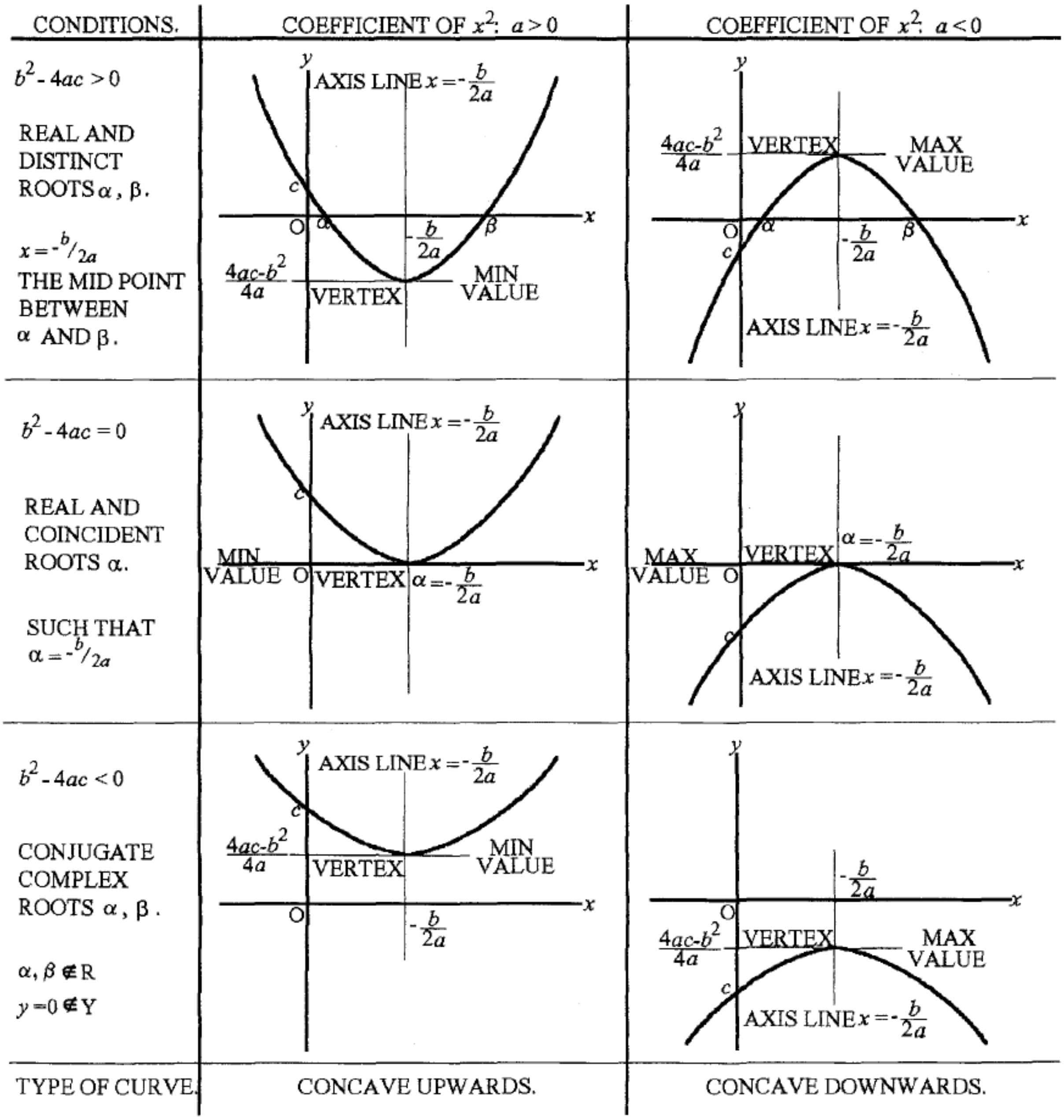
Note that by (16) we have that the minimum value of $f(x)$ i.e. $-{}^{13}/_{4}$, occurs at $x = -{}^{3}/_{2}$.

The symmetry of quadratic curve

Putting all of the information together we have the following quotable facts in relation to the quadratic function:

$$y = ax^2 + bx + c,\ x \in \mathrm{R} \qquad (1)$$

and its curve.



THE SIX FUNDAMENTAL FORMS OF QUADRATIC CURVE.