On the basis of the *extended definition* of a logarithm, which we are assuming to hold, common and natural logarithms are currently respectively denoted by:

$$\lg s, \qquad \ln s \tag{3}$$

Except where otherwise required, it is this notation that we shall adopt.

There are *tables* for common and natural logarithms, and usually these are the logarithms that appear on most *mathematical or scientific calculators*.

Much more will be said at a later stage with regard to logarithms, in particular to the base e, when we consider the theory surrounding the idea of a logarithmic function.

Simple Exponential And Logarithmic Equations.

Any equation involving an unknown quantity as an *index*, is referred to as being an *exponential equation*. Thus for example:

$$2^x = 32 \tag{1}$$

is an exponential equation in the unknown x.

Clearly, *by inspection*, the solution to this simple equation is:

$$x = 5 \tag{2}$$

Now consider the exponential equation:

$$25^x = 125 \tag{3}$$

Here it is not immediately clear by inspection as to what is the value of x. however, if we take logarithms to the base 5 on both sides we have that

$$\log_5 25^x = \log_5 125$$

which by the third law of logarithms gives:

$$x \log_5 25 = \log_5 125$$

i.e. gives

$$x = \frac{\log_5 125}{\log_5 25}$$

We thus have, on applying the definition of a logarithm, that

$$x = \frac{3}{2} \tag{4}$$

as the required solution.

When taking logarithms of both sides, as above, any base can be used, provided of course, that the *same base* is used on either side of the equation.

For the given example, 5 was the obvious choice of base; but more often than not, a particular base is not immediately apparent, and when this is so we take logarithms to either base 10 or base e; usually preferring the latter nowadays.

As an instance, consider the equation:

$$2^x = 5 \qquad (5)$$

Here we cannot see the answer by inspection, nor is there a base that allows us to find x without resort to a calculator. This being the case we take logarithms to the base e, and have that on applying the third law

$$x \ln 2 = \ln 5$$

We thus have that by this

$$x = \frac{\ln 5}{\ln 2} = \frac{1.609438}{0.693147}$$

giving as solution:

$$x = 2.32193 \text{ to 5 decimal places} \qquad (6)$$

Note that in using the calculator, the division of ln 5 by ln2 can be carried out without having to write down the individual values of these logarithms, and then again use the calculator to perform the division.

Any equation involving the *logarithm* of an unknown quantity is referred to as a *logarithmic equation*. As an example of a simple form of this type of equation consider:

$$\log_2 x = 3 \qquad (7)$$

To solve this, we have by definition that given (7), then

$$x = 2^3$$

i.e.

$$x = 8 \qquad (8)$$

a result that here, might have been seen by inspection.

As another basic example, consider

$$\lg x = 1.5 \qquad (9)$$

By definition we have that now

$$x = 10^{1.5}$$

Using the calculator to evaluate the right hand side we have that

$$x = 31.62278 \text{ to 5 decimal places} \qquad (10)$$

We note that the solutions of (7) and (9) respectively amount to finding the *antilogarithms* of 3 to the base 2, and 1.5 to the base 10; the latter of which can be carried out *directly*, by using the *inverse, or shift* facility on the calculator.

More complex versions of exponential and logarithmic equations will be dealt with later, including the case where the unknown quantity may appear as the base of a logarithm.

For now we point out that no matter what the form of equation, at some stage the definition and laws of logarithms will need to be applied, and great care should be taken in doing this, especially in relation to the basic laws, which are often, through inexperience, mis-applied.

LOGARITHMIC AND EXPONENTIAL EQUATIONS.

Logarithmic Equations.

Any equation involving the logarithm of an unknown quantity is referred to as a logarithmic equation. Thus as a very basic example:

$\log_2 x = 5$ (1)

is such an equation. By *definition* of a logarithm we have for this equation that:

$x = 2^5$

i.e. that:

$x = 32$ (2)

is the required solution.

In a similar manner we have that given the equation:

$\ln x = 5$ (3)

then by definition:

$x = e^5$

which from the tables, or by calculator, gives:

$x = 148.41$ to 2 decimal places (4)

Note that the solutions to (1) and (3) respectively amount to finding the *antilogarithms* of 5 to the base 2 and 5 to the base e, the latter of which can be carried out *directly* by using the *inverse* or *shift* facility on a calculator.

Of a less simple nature is an equation of the form:

$2\ln x - \ln 2x = 1$

Here we first apply the laws related to logarithms and have that we may write the equation as:

$\ln x^2 - \ln 2x = 1$ (5)

i.e. as:

$\ln \frac{x^2}{2x} = 1$

giving:

$\ln \frac{x}{2} = 1$

We now have that by definition:

$$\frac{x}{2} = e^1$$

i.e.

$$x = 2e$$

giving:

$x = 5.437$ to 3 decimal places. (6)

Now consider an equation of the form:

$$\ln x + \lg x = 1 \qquad (7)$$

Here we have the logarithms to a different base, and must first express the equation in terms of one base only. Applying the change of base formula we have that we may write the equation as:

$$\ln x + \frac{\ln x}{\ln 10} = 1$$

and hence have that:

$$2.3026 \ln x + \ln x = 2.3026$$

giving:

$$\ln x = \frac{2.3026}{1 + 2.3026}$$

i.e.

$$\ln x = 0.6972$$

Hence by definition we have that:

$$x = e^{0.6972}$$

i.e.

$x = 2.0081$ to 4 decimal places. (8)

Again note, we may find the antilogarithm x of 0.6972 to the base e by using the inverse facility on a calculator.

A variation on the type of equation with more than one base is that where the unknown itself may be a base. Consider for instance the equation:

$$\ln x - 2\log_x e = 1 \qquad (9)$$

Here by the change of base formula we have that:

$$\ln x - \frac{2}{\ln x} = 1$$

We now have that, on rearranging:

$$(\ln x)^2 - \ln x - 2 = 0 \quad (10)$$

which is a *quadratic equation* in terms of ln x. On factorising we have that:

$$(\ln x + 1)(\ln x - 2) = 0$$

giving:

$$\ln x = -1,\ \ln x = 2$$

from which we have that:

$$x = e^{-1},\ x = e^2$$

i.e.

$$x = 0.3679,\ x = 7.3891 \text{ to 4 decimal places} \quad (11)$$

As a final example, consider the *simultaneous equations*:

$$\begin{aligned} \ln x + \ln y &= 1 \\ 2\ln x - \ln y &= -1 \end{aligned} \quad (12)$$

Here we simply apply the usual techniques for such equations, regarding ln x and ln y as being the unknowns. We have that on adding the equations:

$$3\ln x = 0$$

i.e.

$$\ln x = 0 \quad (13)$$

giving:

$$x = 1 \quad (14)$$

Substituting from (13) into the first of the equations in (12) we have that:

$$\ln y = 1$$

and hence have that:

$$y = e \quad (15)$$

We note that we can take an *alternative approach* to the solution of equation (12) by applying the basic laws to each of them, giving $\ln(xy) = 1$, and $\ln({}^{x^2}/_y) = -1$, which in turn gives $xy = e$, and ${}^{x^2}/_y = {}^1/_e$, from which we can readily determine x and y.

Exponential Equations.

Any equation involving an unknown quantity as an index, or exponent, is generally referred to as being an exponential equation. Thus for example:

$3^x = 27$ (1)

is an exponential equation. Clearly the solution to this equation is, *by inspection*:

$x = 3$ (2)

Now consider the equation:

$3^x = 5$ (3)

Here the solution cannot be seen by inspection. We can however for this relatively simple type of equation, take the logarithm of each side, to say, the base e i.e. we can write that:

$\ln 3^x = \ln 5$

and therefore that:

$x\ln 3 = \ln 5$

We thus have that:

$$x = \frac{\ln 5}{\ln 3}$$

giving:

$x = 1.4650$ to 4 decimal places. (4)

Note that the logarithms can also be taken for equation (1), where the obvious base to choose is 3, giving:

$x\log_3 3 = \log_3 27$

i.e.

$x = 3$

as before.

Generally when it is not clear as to a particular base being applicable, either logarithms to the base e or 10 are used.

It is important to realise however that exponential equations in general, do not lend themselves to taking logarithms at the outset as a method of solution. A typical example of one that doesn't is:

$5^{2x} - 5^{x+1} + 4 = 0$ (5)

Here it is a question of recognizing that the equation is *quadratic in form*, since we may write it as:

$$(5^x)^2 - 5(5^x) + 4 = 0$$

and regard the unknown as being 5^x. On factorising we have that:

$$(5^x - 1)(5^x - 4) = 0$$

giving:

$$5^x = 1,\ 5^x = 4$$

Clearly the only solution to $5^x = 1$ is:

$$x = 0 \qquad (6)$$

For the second of the solutions i.e. for $5^x = 4$, we have on taking logarithms to the base 10, that:

$$x\lg 5 = \lg 4$$

i.e.

$$x = \frac{\lg 4}{\lg 5}$$

giving the other solution as:

$$x = 0.8613 \text{ to 4 decimal places} \qquad (7)$$

As for the logarithmic equations, we may also have *simultaneous exponential equations*. As an example consider the equations below in the two unknowns x and y:

$$\begin{aligned} 3^{(x+3)} &= 9^{(2-y)} \\ 9^{(x-4)} &= 3^{-y} \end{aligned} \qquad (8)$$

Taking logarithms to a base 3 we have the equations:

$$\begin{aligned} (x+3)\log_3 3 &= (2-y)\log_3 9 \\ (x-4)\log_3 9 &= -y\log_3 3 \end{aligned}$$

i.e. the equations:

$$\begin{aligned} x+3 &= 2(2-y) \\ 2(x-4) &= -y \end{aligned}$$

giving the linear simultaneous equations:

$$\begin{aligned} x+2y &= 1 \\ 2x+y &= 8 \end{aligned}$$

The solutions to which are:

$$x = 5,\ y = -2 \qquad (9)$$