THEORY OF LOGARITHMS.

Introduction.

Let a be any real number, and q any rational number; then by the theory of indices, a^q is *well-defined*, in that for each given value of a and q, it leads to either *a unique real, or complex number.*

If we restrict a to being a *positive number,* it is clear that then, for all rational numbers q, a^q is always *real* and in particular, *positive.*

In other words, for a given $a \in R^+$, there corresponds to each $q \in Q$, a unique real number $p \in R^+$, such that:

$$p = a^q \qquad (1)$$

Except for the trivial case when $a = 1$, which leads to $p = 1$ for all values of $q \in Q$, there is generated in relation to each given positive value of a, as q takes on all possible rational values, *an infinite set of distinct values of* $p \in R^+$.

It is important to realise however, that these positive values of p are only those positive numbers which for a given $a \neq 1$, relate to q being *rational.* That is to say, for any given positive value of $a \neq 1$, as q takes on all possible rational values, so *the values of p form only a particular infinite set of positive numbers.*

The implication of this is that, *given any positive number p and a positive value of* $a \neq 1$, *then there does not necessarily exist a rational number q such that* (1) *holds.*

However, as was noted in the theory of indices, later, elsewhere, the concept of a power of a number is re-defined, and when a is positive, a meaning is attached to it being raised to an *irrational,* as well as rational power.

With this extension of the concept it is shown that *for any given positive value of* $a \neq 1$, *and real number r,* a^r *takes on distinctly, all possible positive numbers, as r takes on all possible real values.*

The implication now, is that *given any positive number s, and a positive value of* $a \neq 1$, *then there always exists a unique real number r such that:*

$$s = a^r \qquad (2)$$

It is upon (1) that the theory of logarithms is based in algebra, and by acceptance of (2), extended so as to apply more generally to all positive numbers.

The Algebraic Definition Of A Logarithm.

Let $a \neq 1$ be a given *positive number,* and let q be any *rational number,* then as stated in the introduction, there corresponds to each value of q *a positive number p* such that:

$$p = a^q \qquad (1)$$

We in algebra, define the rational number q to be the *logarithm* of the positive number p, in respect of the given positive number $a \neq 1$, referred to as the *base a*.

To express this fact we write that:

$$q = \log_a p \qquad (2)$$

We immediately note that on this basis of definition, the logarithm of a number is only defined for positive numbers p, and then very restrictively, only for those positive numbers which for a given base $a \neq 1$, give rise to the logarithm q being *rational.*

The Extension Of The Definition.

By accepting the fact that when $a \neq 1$ is *a given positive number*, and r is *any real number*, then a^r takes on distinctly all possible positive numbers as r takes on all possible real values, we have on taking r to be the logarithm of s to the base a in $s = a^r$, a definition of a logarithm which applies to *all* positive numbers s, rather than a restricted set of such numbers.

We note that even on the basis of the extended definition, the concept of the logarithm of a number, no matter what the positive number base $a \neq 1$, applies only to *positive numbers s*. All negative numbers and zero, are excluded from the definition.

We also note that the concept only applies to *a positive base a*; with the exclusion of $a = 1$, for the reason that this relates to only one positive number s, namely $s = 1$.

Clearly among all of the logarithms of s to the base a, are included the logarithms of p to the same base, which have only rational values, the rational numbers being a sub-set of the real numbers.

In summary then; given that

$$s = a^r \qquad (1)$$

we define the real number index r when a is positive and not equal to unity, to be the logarithm of the positive number s to the base a, and indicate this fact by writing:

$$r = \log_a s \qquad (2)$$

The justification for the extension, as already mentioned, will be dealt with later elsewhere.

The theory which follows will be based on the extended definition, and further, on the acceptance of the fact that the laws of indices can be shown to apply, not only to the case when the index is rational, but also when it is a real number, provided that then, the number which is being raised to the power is *positive*.

Some First Results.

A question which we may ask, and which may possibly determine the logarithm of a given positive number s to a given positive number base $a \neq 1$, is: '*to what power r do we have to raise the base a, in order that it equals the number s*?'

Applying this question to the case where $a \neq 1$ is any given positive base, and $s = 1$, we immediately have the answer that $r = 0$ i.e. for *any base a*:

$$\log_a 1 = 0 \qquad (1)$$

Likewise, on posing the question for the particular case when $s = a$, we see that, no matter what the positive base a,

$$\log_a a = 1 \qquad (2)$$

As will be found, only in simple cases can the asking of the question determine the logarithm of a number. More often than not, we can only readily ascertain the value by recourse to *standard tables*, or through the use of a suitable *calculator*. Both of these however, are restricted to two given bases, and for other bases we need to first apply what is known as *the change of base formula,* with which we shall deal later.

The values of the logarithms of numbers to the two given bases, as held in the tables and calculator, are determined by way of *power series*, that can be structured to represent the logarithms of the numbers to the given bases. Suffice it to say here, that aspects of such series are dealt with later in the course.

As an instance of the construction of a logarithmic table by posing the basic question given above, we have the following results in relation to logarithms to the base 2, for some of the numbers to which the question is applicable.

number s	$\log_2 s$	number s	$\log_2 s$	number s	$\log_2 s$
+1/64	-6	+1/4	-2	+4	+2
+1/32	-5	+1/2	-1	+8	+3
+1/16	-4	+1	0	+16	+4
+1/8	-3	+2	+1	+32	+5

From the table it is clear that for all $1 < s < \infty$, $\log_2 s > 0$; and for all $0 < s < 1$, $\log_2 s < 0$. This is so for all bases $a > 1$. On the other hand, the converse is true when $0 < a < 1$.

It is left as an exercise to demonstrate this latter fact, by finding the logarithms of the same numbers as above, to the base $^1/_2$.

Returning to the case of logarithms to the base 2, we see that even for whole numbers such as 3, 5, 6,..., we cannot determine their logarithms to the given base by asking the question. Here we need to apply the change of base formula, and make use of the calculator. As can be confirmed later, it follows on this basis that to 5 decimal places:

$\log_2 3 = 1.58496$, $\quad \log_2 5 = 2.32193$, $\quad \log_2 6 = 2.58496$

These values are in the order of what might be expected on referring to the table, where it is seen that for $a = 2$, the values r of the logarithm, increase for increasing values of the number s.

This is characteristic for all bases $a > 1$, with the converse being so for all bases $0 < a < 1$.

Fundamental when dealing with the basic aspects of logarithms, other than the posing of the aforementioned question, is the fact that if we are given that s and a are positive numbers, $a \neq 1$, then *by definition we may change between the statements*:

$$s = a^r \tag{3}$$

and

$$r = \log_a s \tag{4}$$

as required.

In other words, given (3) we may write (4), and vice versa. This interchange will in particular be applied in developing the laws of logarithms.

Firstly however, we note that by (3) and (4) we may write that

$$s \equiv a^{\log_a s} \tag{5}$$

The writing of s in terms of a power of a number a i.e. as what is called *an exponent* of a, can prove useful on occasion when manipulating expressions that also involve other exponents of a.

Raising both sides of (5) to the power t (say), where t is any real number, we also see by the extended laws of indices that:

$$s^t \equiv a^{t\log_a s} \tag{6}$$

a result which again can prove useful from time to time.

The Logarithmic Laws.

We now establish three important laws in connection with logarithms.

Throughout we take it that $a \neq 1$ is a *positive number*, that the indicated values of s are also *positive numbers*, and that those of r are *real numbers*.

On this basis let:

$$r_1 = \log_a s_1, \quad r_2 = \log_a s_2$$

then by definition we have that

$$s_1 = a^{r_1}, \quad s_2 = a^{r_2}$$

It follows that

$$s_1.s_2 = a^{(r_1+r_2)}$$

on applying the extended laws of indices.

But again by the definition, this implies that:

$$r_1 + r_2 = \log_a (s_1.s_2)$$

In other words we have that

$$\log_a(s_1.s_2) = \log_a s_1 + \log_a s_2 \qquad (1)$$

Likewise by the extended laws of indices, we have that

$$\frac{s_1}{s_2} = a^{(r_1 - r_2)}$$

and hence by definition that

$$r_1 - r_2 = \log_a\left(\frac{s_1}{s_2}\right)$$

giving

$$\log_a\left(\frac{s_1}{s_2}\right) = \log_a s_1 - \log_a s_2 \qquad (2)$$

Now suppose that

$$r = \log_a s$$

then by definition we have that

$$s = a^r$$

and given that t is any real number, by the extended laws of indices that

$$s^t = a^{rt}$$

From this we have, once again by definition, that:

$$rt = \log_a(s^t)$$

i.e. we have that

$$\log_a(s^t) = t\log_a s \qquad (3)$$

Results (1), (2) and (3) represent the three fundamental laws of logarithms, and are of great importance in respect of their manipulation.

They also give rise to methods where by the multiplication, division, and raising to a power of numbers, can be carried out through the use of logarithms, though somewhat redundant these days, in the age of the calculator!

Nevertheless it is worth mentioning that for example to multiply two numbers s_1 and s_2, we can simply take the logarithms of the numbers to a given base, *add* one logarithm to the other, to determine the logarithm of the product $s_1.s_2$, and then from this, in a process known as finding the *antilogarithm*, determine $s_1.s_2$ itself. We see that in effect, multiplication becomes a matter of addition.

In a similar way, division is reduced to the more simple operation of subtraction; and that of raising a power to the operation of multiplication. Actually, before the advent of the calculator, it was to these latter two aspects of numerical calculation that the logarithmic methods were most applicable.

As a basic example in relation to *the manipulation of logarithms*, consider the finding, without the use of a calculator, of:

$\log_{10} 5 + \log_{10} 20$ (4)

By (1) we immediately have that we may write that

$\log_{10} 5 + \log_{10} 20 = \log_{10}(5 \times 20)$

and hence have that

$\log_{10} 5 + \log_{10} 20 = \log_{10} 100$

Now clearly we must raise 10 to the power of 2 in order that it equals 100, and hence $\log_{10} 100 = 2$ i.e.

$\log_{10} 5 + \log_{10} 20 = 2$ (5)

As another example consider

$\log_2 20 - \log_2 5$ (6)

Here by (2) we have that

$\log_2 20 - \log_2 5 = \log_2 \frac{20}{5} = \log_2 4$

i.e.

$\log_2 20 - \log_2 5 = 2$ (7)

Now suppose that we require to find, without recourse to a calculator

$\frac{1}{3}\log_3 27$ (8)

Here, on applying (3) we have that

$\frac{1}{3}\log_3 27 = \log_3(27^{1/3}) = \log_3 3$

i.e.

$\frac{1}{3}\log_3 27 = 1$ (9)

In fact we may take an alternative approach in this case since clearly

$\log_3 27 = 3$

and hence

$\frac{1}{3}\log_3 27 = \frac{1}{3}.3 = 1$

as before.

Laws (1), (2) and (3) can be used in combination with each other. As an instance of this consider

$$\frac{1}{2}\log_{10}9+\log_{10}250-\log_{10}75 \tag{10}$$

Applying (3) first, and then (1) and (2) we get that

$$\frac{1}{2}\log_{10}9+\log_{10}250-\log_{10}75=\log_{10}9^{1/2}+\log_{10}250-\log_{10}75$$

$$=\log_{10}\frac{9^{1/2}\times 250}{75}$$

$$=\log_{10}10$$

i.e.

$$\frac{1}{2}\log_{10}9+\log_{10}250-\log_{10}75=1 \tag{11}$$

As a final example we find the value of

$$\frac{\log_a 64}{\log_a 2} \tag{12}$$

Here we have that

$$\frac{\log_a 64}{\log_a 2}=\frac{\log_a 2^6}{\log_a 2}=\frac{6\log_a 2}{\log_a 2}=6 \tag{13}$$

Note that the laws (1), (2) and (3) *only apply when the base is the same throughout.*

Change Of Base.

Suppose that we require to find the logarithm of a number s to a base a, for which tables, or a suitable calculator are not available; but on the other hand, *are*, in respect of the logarithm of s to another base b.

We establish a formula whereby we can change from one base to the other, and in consequence, are then in a position to determine the logarithm of s to the base a.

Taking

$r=\log_a s$

we have by definition that then

$s=a^r$

Now clearly, *for any equality we may take the logarithm of both sides, to the same base, and maintain the equality.*

Doing so to the base b in respect of the latter equality we have that

$\log_b s=\log_b(a^r)$

It follows from this that

$\log_b s = r \log_b a$

i.e. that

$$r = \frac{\log_b s}{\log_b a}$$

In other words we have that

$$\log_a s = \frac{\log_b s}{\log_b a} \qquad (1)$$

This is known as the *change of base formula.*

As an example of its application to the calculation of a logarithm consider the finding of

$$\log_2 3 \qquad (2)$$

Available on a calculator are logarithms of numbers to a base 10, and by (1) we may write that

$$\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2}$$

Using the calculator we have that

$$\log_2 3 = \frac{0.477121}{0.301030}$$

giving

$$\log_2 3 = 1.58496 \text{ to 5 decimal places.} \qquad (3)$$

as given earlier.

As another instance of the application of the formula consider the evaluation of:

$$\log_2 20 - \log_4 25 \qquad (4)$$

Here we note, the bases being different, the laws of logarithms *cannot* be applied. However, by using the change of base formula we have that

$$\log_2 20 - \log_4 25 = \log_2 20 - \frac{\log_2 25}{\log_2 4}$$

giving

$$\log_2 20 - \log_4 25 = \log_2 20 - \frac{1}{2}\log_2 25$$

We are now in a position where the laws can be applied and have that

$$\log_2 20 - \log_4 25 = \log_2 20 - \log_2 25^{1/2}$$

$$= \log_2 20 - \log_2 5$$

$$= \log_2 \frac{20}{5}$$

$$= \log_2 4$$

i.e.

$$\log_2 20 - \log_4 25 = 2 \qquad (5)$$

The formula is also used in the general manipulation of logarithms where different bases are involved, and in this respect a particular quotable version of it can prove useful. Taking $s = b$ in (1) we obtain

$$\log_a b = \frac{\log_b b}{\log_b a}$$

i.e. we have that

$$\log_a b = \frac{1}{\log_b a} \qquad (6)$$

This is sometimes referred to as the *inverse rule* in respect of bases.

Common And Natural Logarithms.

The logarithms that were primarily used for arithmetic calculations were those to a base 10. Such logarithms are referred to as *common logarithms*, and were introduced by Briggs in 1615.

He was a contemporary of Napier, the inventor of logarithms, and whose name is associated with logarithms to the very important base e, where e is an irrational number, and has value 2.71828 to 5 decimal places.

Besides being known as Naperian logarithms, logarithms to the base e are also referred to as Hyperbolic logarithms, but more often than not, they are now referred to as being *natural logarithms*.

With regard to notation; based on a *strictly algebraic definition*, the common and natural logarithms of a number s, are respectively denoted by:

$$\log_{10} s, \qquad \log_e s \qquad (1)$$

However, in many of the older textbooks it will be found that simply:

$$\log s \qquad (2)$$

is written, but it is usually clear from the context as to which of the two bases this refers.

On the basis of the *extended definition* of a logarithm, which we are assuming to hold, common and natural logarithms are currently respectively denoted by:

$\lg s, \qquad \ln s$ (3)

Except where otherwise required, it is this notation that we shall adopt.

There are *tables* for common and natural logarithms, and usually these are the logarithms that appear on most *mathematical or scientific calculators*.

Much more will be said at a later stage with regard to logarithms, in particular to the base e, when we consider the theory surrounding the idea of a logarithmic function.

Simple Exponential And Logarithmic Equations.

Any equation involving an unknown quantity as an *index*, is referred to as being an *exponential equation*. Thus for example:

$2^x = 32$ (1)

is an exponential equation in the unknown x.

Clearly, *by inspection*, the solution to this simple equation is:

$x = 5$ (2)

Now consider the exponential equation:

$25^x = 125$ (3)

Here it is not immediately clear by inspection as to what is the value of x. however, if we take logarithms to the base 5 on both sides we have that

$$\log_5 25^x = \log_5 125$$

which by the third law of logarithms gives:

$$x \log_5 25 = \log_5 125$$

i.e. gives

$$x = \frac{\log_5 125}{\log_5 25}$$

We thus have, on applying the definition of a logarithm, that

$x = \frac{3}{2}$ (4)

as the required solution.

When taking logarithms of both sides, as above, any base can be used, provided of course, that the *same base* is used on either side of the equation.

For the given example, 5 was the obvious choice of base; but more often than not, a particular base is not immediately apparent, and when this is so we take logarithms to either base 10 or base e; usually preferring the latter nowadays.

As an instance, consider the equation:

$$2^x = 5 \qquad (5)$$

Here we cannot see the answer by inspection, nor is there a base that allows us to find x without resort to a calculator. This being the case we take logarithms to the base e, and have that on applying the third law

$$x \ln 2 = \ln 5$$

We thus have that by this

$$x = \frac{\ln 5}{\ln 2} = \frac{1.609438}{0.693147}$$

giving as solution:

$$x = 2.32193 \text{ to 5 decimal places} \qquad (6)$$

Note that in using the calculator, the division of ln 5 by ln2 can be carried out without having to write down the individual values of these logarithms, and then again use the calculator to perform the division.

Any equation involving the *logarithm* of an unknown quantity is referred to as a *logarithmic equation*. As an example of a simple form of this type of equation consider:

$$\log_2 x = 3 \qquad (7)$$

To solve this, we have by definition that given (7), then

$$x = 2^3$$

i.e.

$$x = 8 \qquad (8)$$

a result that here, might have been seen by inspection.

As another basic example, consider

$$\lg x = 1.5 \qquad (9)$$

By definition we have that now

$$x = 10^{1.5}$$

Using the calculator to evaluate the right hand side we have that

$$x = 31.62278 \text{ to 5 decimal places} \qquad (10)$$

We note that the solutions of (7) and (9) respectively amount to finding the *antilogarithms* of 3 to the base 2, and 1.5 to the base 10; the latter of which can be carried out *directly*, by using the *inverse, or shift* facility on the calculator.

More complex versions of exponential and logarithmic equations will be dealt with later, including the case where the unknown quantity may appear as the base of a logarithm.

For now we point out that no matter what the form of equation, at some stage the definition and laws of logarithms will need to be applied, and great care should be taken in doing this, especially in relation to the basic laws, which are often, through inexperience, mis-applied.

LOGARITHMIC AND EXPONENTIAL EQUATIONS.

Logarithmic Equations.

Any equation involving the logarithm of an unknown quantity is referred to as a logarithmic equation. Thus as a very basic example:

$$\log_2 x = 5 \qquad (1)$$

is such an equation. By *definition* of a logarithm we have for this equation that:

$$x = 2^5$$

i.e. that:

$$x = 32 \qquad (2)$$

is the required solution.

In a similar manner we have that given the equation:

$$\ln x = 5 \qquad (3)$$

then by definition:

$$x = e^5$$

which from the tables, or by calculator, gives:

$$x = 148.41 \text{ to 2 decimal places} \qquad (4)$$

Note that the solutions to (1) and (3) respectively amount to finding the *antilogarithms* of 5 to the base 2 and 5 to the base e, the latter of which can be carried out *directly* by using the *inverse* or *shift* facility on a calculator.

Of a less simple nature is an equation of the form:

$$2\ln x - \ln 2x = 1$$

Here we first apply the laws related to logarithms and have that we may write the equation as:

$$\ln x^2 - \ln 2x = 1 \qquad (5)$$

i.e. as:

$$\ln \frac{x^2}{2x} = 1$$

giving:

$$\ln \frac{x}{2} = 1$$

We now have that by definition:

$\frac{x}{2} = e^1$

i.e.

$x = 2e$

giving:

$x = 5.437$ to 3 decimal places. (6)

Now consider an equation of the form:

$\ln x + \lg x = 1$ (7)

Here we have the logarithms to a different base, and must first express the equation in terms of one base only. Applying the change of base formula we have that we may write the equation as:

$\ln x + \frac{\ln x}{\ln 10} = 1$

and hence have that:

$2.3026 \ln x + \ln x = 2.3026$

giving:

$\ln x = \frac{2.3026}{1 + 2.3026}$

i.e.

$\ln x = 0.6972$

Hence by definition we have that:

$x = e^{0.6972}$

i.e.

$x = 2.0081$ to 4 decimal places. (8)

Again note, we may find the antilogarithm x of 0.6972 to the base e by using the inverse facility on a calculator.

A variation on the type of equation with more than one base is that where the unknown itself may be a base. Consider for instance the equation:

$\ln x - 2\log_x e = 1$ (9)

Here by the change of base formula we have that:

$\ln x - \frac{2}{\ln x} = 1$

We now have that, on rearranging:

$$(\ln x)^2 - \ln x - 2 = 0 \tag{10}$$

which is a *quadratic equation* in terms of ln x. On factorising we have that:

$$(\ln x + 1)(\ln x - 2) = 0$$

giving:

$$\ln x = -1,\ \ln x = 2$$

from which we have that:

$$x = \mathrm{e}^{-1},\ x = \mathrm{e}^2$$

i.e.

$$x = 0.3679,\ x = 7.3891 \text{ to 4 decimal places} \tag{11}$$

As a final example, consider the *simultaneous equations*:

$$\begin{aligned} \ln x + \ln y &= 1 \\ 2\ln x - \ln y &= -1 \end{aligned} \tag{12}$$

Here we simply apply the usual techniques for such equations, regarding ln x and ln y as being the unknowns. We have that on adding the equations:

$$3\ln x = 0$$

i.e.

$$\ln x = 0 \tag{13}$$

giving:

$$x = 1 \tag{14}$$

Substituting from (13) into the first of the equations in (12) we have that:

$$\ln y = 1$$

and hence have that:

$$y = \mathrm{e} \tag{15}$$

We note that we can take an *alternative approach* to the solution of equation (12) by applying the basic laws to each of them, giving $\ln(xy) = 1$, and $\ln({}^{x^2}/_{y}) = -1$, which in turn gives $xy = \mathrm{e}$, and ${}^{x^2}/_{y} = {}^{1}/_{\mathrm{e}}$, from which we can readily determine x and y.

Exponential Equations.

Any equation involving an unknown quantity as an index, or exponent, is generally referred to as being an exponential equation. Thus for example:

$3^x = 27$ (1)

is an exponential equation. Clearly the solution to this equation is, *by inspection*:

$x = 3$ (2)

Now consider the equation:

$3^x = 5$ (3)

Here the solution cannot be seen by inspection. We can however for this relatively simple type of equation, take the logarithm of each side, to say, the base e i.e. we can write that:

$\ln 3^x = \ln 5$

and therefore that:

$x\ln 3 = \ln 5$

We thus have that:

$$x = \frac{\ln 5}{\ln 3}$$

giving:

$x = 1.4650$ to 4 decimal places. (4)

Note that the logarithms can also be taken for equation (1), where the obvious base to choose is 3, giving:

$x\log_3 3 = \log_3 27$

i.e.

$x = 3$

as before.

Generally when it is not clear as to a particular base being applicable, either logarithms to the base e or 10 are used.

It is important to realise however that exponential equations in general, do not lend themselves to taking logarithms at the outset as a method of solution. A typical example of one that doesn't is:

$5^{2x} - 5^{x+1} + 4 = 0$ (5)

Here it is a question of recognizing that the equation is *quadratic in form*, since we may write it as:

$$(5^x)^2 - 5(5^x) + 4 = 0$$

and regard the unknown as being 5^x. On factorising we have that:

$$(5^x - 1)(5^x - 4) = 0$$

giving:

$$5^x = 1, \; 5^x = 4$$

Clearly the only solution to $5^x = 1$ is:

$$x = 0 \qquad (6)$$

For the second of the solutions i.e. for $5^x = 4$, we have on taking logarithms to the base 10, that:

$$x\lg 5 = \lg 4$$

i.e.

$$x = \frac{\lg 4}{\lg 5}$$

giving the other solution as:

$$x = 0.8613 \text{ to 4 decimal places} \qquad (7)$$

As for the logarithmic equations, we may also have *simultaneous exponential equations*. As an example consider the equations below in the two unknowns x and y:

$$\begin{aligned} 3^{(x+3)} &= 9^{(2-y)} \\ 9^{(x-4)} &= 3^{-y} \end{aligned} \qquad (8)$$

Taking logarithms to a base 3 we have the equations:

$$\begin{aligned} (x+3)\log_3 3 &= (2-y)\log_3 9 \\ (x-4)\log_3 9 &= -y\log_3 3 \end{aligned}$$

i.e. the equations:

$$\begin{aligned} x+3 &= 2(2-y) \\ 2(x-4) &= -y \end{aligned}$$

giving the linear simultaneous equations:

$$\begin{aligned} x + 2y &= 1 \\ 2x + y &= 8 \end{aligned}$$

The solutions to which are:

$$x = 5, \; y = -2 \qquad (9)$$