January Examination Period 2023-24

ECN115—Mathematical Methods in Economics and Finance ***Duration: 2 hours***

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Answer ALL questions

Calculators are not permitted in this examination. Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner: Evgenii Safonov

Question 1 [16 marks]

Let A, B be sets.

a) Explain/define the notion of set difference $A \backslash B$.

[5 marks]

b) Are there non-empty sets of natural numbers A, B such that $A \backslash B = A$? If yes, give an example of such sets A and B; if no, argue why.

[3 marks]

c) Are there non-empty sets of natural numbers A, B such that $A \backslash B = B$? If yes, give an example of such sets A and B; if no, argue why.

[3 marks]

d) Find all sets of natural numbers D that satisfy the following condition:

$$\{1, 2\} \cup D = \{1, 2, 3\}$$

[5 marks]

Question 2 [17 marks]

Consider the sequences a_n, c_n given by

$$a_n = \frac{n-1}{n^2+1}, \qquad c_n = \frac{n^2-1}{n^2+1}.$$

Assume that b_n is a sequence that satisfies

$$a_n \leq b_n \leq c_n \tag{1}$$

for all $n = 1, 2, ...$

a) Find the limits of sequences a_n, c_n if they exist.

[6 marks]

b) Does there exist a sequence b_n that satisfies eq. (1) and has a finite limit? If yes, provide an example of such sequence, if no, explain why.

[5 marks]

c) Is it true that any sequence b_n that satisfies eq. (1) has a finite limit? If yes, show why it is true (for instance, refer to a result from the class), if no, provide a (counter) example.

[6 marks]

Question 3 [32 marks]

Consider function $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{x} + ax + x^3$$

where $a \in \mathbb{R}$ is a parameter.

a) Calculate the derivative of the function f.

[4 marks]

b) Find the values of x for which $f'(x) = 0$, for which $f'(x) < 0$, and for which $f'(x) > 0$. You answer should depend on the parameter a.

Hint: to solve this question, express the derivative of the function f as a fraction of two polynomials; thus, $f'(x) = \frac{P(x)}{Q(x)}$. Then, find when each of the polynomials P, Q takes positive, negative, and zero values. You can introduce an auxiliary variable y such that P becomes a quadratic polynomial with respect to y.

[15 marks]

c) Let $c > 0$ be a parameter. Is it true that for all values of the parameters $a \in \mathbb{R}$ and $c \in (0, \infty)$, there exists a minimum of the function f over the set $(0, c]$? Can you use the Weierstrass theorem in your analysis?

[6 marks]

d) Using your analysis in (b), (c), find the minimum of the function f over the interval $(0, c]$ when it exists and points at which it is attained. In other words, find

$$\min_{x \in (0,c]} f(x), \qquad \arg\min_{x \in (0,c]} f(x).$$

Your answer should, in general, depend on the parameters $a \in \mathbb{R}$ and $c \in (0, \infty)$. There is no need to simplify your answer if the expression is bulky.

[7 marks]

Question 4 [23 marks]

Consider function $f : [0, \infty) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sqrt{x} & if \quad 0 \le x \le 4 \\ x^{\alpha} & if \quad x > 4 \end{cases}$$

where $\alpha \in \mathbb{R}$ is a parameter.

a) Find all values of the parameter α for which the function f is continuous. Provide a short explanation.

[3 marks]

b) Calculate the following definite (Riemann) integral:

$$\int_0^b f(x)dx$$

where $b > 0$ is a parameter. You can take as given that the integral exists for all considered values of the parameters α and b. Your answer should depend on the parameters α and b.

[15 marks]

c) Use your answer from (b) to find all values of the parameter α for which the following improper integral converges and calculate it when it converges (your answer should depend on the parameter α):

$$\int_0^\infty f(x)dx$$

[5 marks]

Question 5 [12 marks]

Let f, g be functions of the two variables x, y with the domain $\mathbb{R}^2$ given by the formulas below. Find partial derivatives of these functions with respect to variables x, y at point $(x, y) = (1, -2)$.

a) $f(x, y) = 3 + x$.

[4 marks]

b) $g(x, y) = e^{xy^2}$.

[8 marks]

End of Paper