Queen Mary University of London

Main Examination period 2023 – May/June – Semester B

MTH6142: Complex Networks

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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Question 1 [50 marks].
Consider the undirected network $G = (V, E)$ with $N = 15$ nodes described by the following set of links:

$$E = \{(1,2),(1,3),(2,4),(2,5),(3,6),(3,7),(4,8),(4,9),$$
$$(5,10),(5,11),(6,12),(6,13),(7,14),(7,15)\}.$$

(a) Calculate the number of links L in the network, and draw the network. Is the network connected? Which type of network is this? (*Explain your answer*). **[7]**

(b) Write down the degree sequence, the degree distribution and the average degree of the network G. **[7]**

(c) Compute the diameter D of the network G, and list all the pairs of nodes which are at distance D (*Hint: it is not necessary to compute all the node distances to work out the diameter of the graph*). **[6]**

(d) Evaluate the betweenness centrality for all the nodes in the network G. Use the following definition of betweenness centrality of a node i:

$$b_i = \sum_{r=1}^{N} \sum_{s=1}^{N} \frac{n_{rs}^i}{n_{rs}}$$

where n_{rs} is the number of shortest paths between node r and node s, while n_{rs}^i is the number of shortest paths between node r and node s passing through node i. *Hint: (Notice that, according to this definition, paths starting from node i or ending in node j are also counted).* **[11]**

(e) Which are the nodes with the largest possible values of betweenness centrality in the network, and which are those with the largest values of closeness centrality and of degree centrality? **[9]**

(f) Suppose you can add a link to connect two nodes of the network G to obtain another network G'. List one link choice that will create a triangle in the network G', one link choice that will create a cycle of length 4, and link choice that will create a cycle of length 5. How many are all the possible link choices that will guarantee that the network G' has a cycle of length $\ell = 7$. **[10]**

Question 2 [50 marks].
Consider the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,L}$ and assume that the number of links grows as a function of N, as $L(N) = \alpha N$, where α is a positive real constant (independent on N).

(a) Write down the general expression, as a function of N and α, for the probability of finding a given graph $G = (V, E)$ in the ensemble $\mathbb{G}_{N,L(N)}$. **[8]**

(b) Fix now $N = 5$ and consider the three cases $\alpha_1 = 0.8$, $\alpha_2 = 1$, and $\alpha_3 = 1.2$. Evaluate, in each of the three cases, the probability of finding a given graph $G = (V, E)$ with 5 nodes in the ensemble $\mathbb{G}_{N,L(N)}$. **[6]**

(c) Find a general expression for the average node degree $\langle k \rangle$ in a graph of the ensemble $\mathbb{G}_{N,L(N)}$, and evaluate the average node degree $\langle k \rangle_1$, $\langle k \rangle_2$ and $\langle k \rangle_3$ for the graphs in each of the three ensembles considered in point (b) (i.e. by fixing $N = 5$ and $\alpha_1 = 0.8$, $\alpha_2 = 1$, $\alpha_3 = 1.2$ respectively). **[7]**

(d) Find the function $p(N)$ of the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,p}$ which best approximate the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,L(N)}$. **[5]**

(e) For the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,p(N)}$ found in point (d), write down the distribution $P(L)$ for the total number of links L in a graph. Calculate the variance of such distribution and find an expression for the ratio r between the standard deviation and the expectation value of L. **[8]**

(f) Consider the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,p(N)}$ found in point (d), assume $N \to \infty$, and find the condition on α for the graphs to have a giant component. Find the same result using the Molloy-Reed criterion. **[8]**

(g) Consider the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,p(N)}$ found in point (d), assume $N \to \infty$, and find the expected numbers of triangles and 4-cliques as a function of α. **[8]**

End of Paper.