AR(1) : $Y_t = \phi Y_{t-1} + \varepsilon_t$

$|\phi| < 1$ → ~~NON~~-STATIONARITY

$|\phi| = 1$ → NON-STATIONARITY

$\cancel{\phi > 1}$ → NON-STATIONARITY

AR(1) : $Y_t = Y_{t-1} + \varepsilon_t$ (RANDOM WALK)

$Y_t = \mu + Y_{t-1} + \varepsilon_t$ (RANDOM WALK WITH DRIFT)

$$X_t = \mu + X_{t-1} + \varepsilon_t$$

where $X_0 = 0$

$\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

$$\rightarrow X_t = \mu + X_{t-1} + \varepsilon_t$$

$$= \mu + (\mu + X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t =$$

$$= 2\mu + X_{t-2} + \varepsilon_{t-1} + \varepsilon_t$$

$$= 2\mu + (\mu + X_{t-3} + \varepsilon_{t-2}) + \varepsilon_{t-1} + \varepsilon_t$$

$$= 3\mu + X_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t$$

$$= \ldots$$

$$= t\mu + \underbrace{X_0}_{X_{t-t} = X_0 = 0} + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots + \underbrace{\varepsilon_1}_{= \varepsilon_{t-(t-1)}}$$

$$= t\mu + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots + \varepsilon_1$$

(i) $E(X_t) = ?$

$$E(X_t) = E(t\mu + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots + \varepsilon_1)$$

$$= \underbrace{E(t\mu)}_{=t\mu} + \underbrace{E(\varepsilon_t)}_{=0} + \underbrace{E(\varepsilon_{t-1})}_{=0} + \underbrace{E(\varepsilon_{t-2})}_{=0} + \ldots$$

BECAUSE ε_t IS WN

$E(X_t) = t\mu \longrightarrow$ SERIES IS NOT STATIONARY

(ii) $Var(X_t) = ?$

$$Var(X_t) = E\left[(X_t - E(X_t))^2\right]$$

$$= E\left[\left((t\mu + \varepsilon_t + \varepsilon_{t-1} + \ldots) - t\mu\right)^2\right]$$

$$= E\left[(\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots + \varepsilon_1)^2\right]$$

$$= \underbrace{E(\varepsilon_t^2)}_{=Var(\varepsilon_t)=} + \underbrace{E(\varepsilon_{t-1}^2)}_{\sigma_\varepsilon^2} + \underbrace{E(\varepsilon_{t-2}^2)}_{\sigma_\varepsilon^2} + \ldots$$

$$= E\left[(\varepsilon_t - E(\varepsilon_t))^2\right] = E\left[\varepsilon_t^2\right] = \sigma_\varepsilon^2$$

$$= \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \ldots$$

$= t\sigma_\varepsilon^2 \longrightarrow$ SERIES IS NOT STAT.

iii) $\operatorname{Cov}(X_t, X_s) = ?$

WE KNOW $X_t = \mu + X_{t-1} + \varepsilon_t$

$$X_t = t\mu + \varepsilon_t + \varepsilon_{t-1} + \dots + \varepsilon_1$$

LET'S DEFINE X_s

$$X_s = \mu + X_{s-1} + \varepsilon_t$$

$$X_s = s\mu + \varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1$$

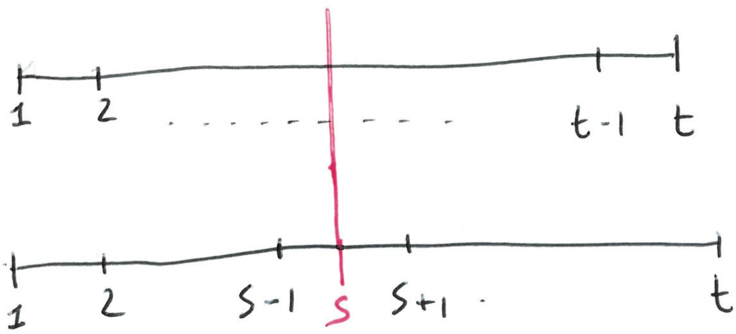
WE ASSUME $s < t$

$\operatorname{Cov}(X_t, X_s) =$

$$= E\left[(X_t - E(X_t))\,(X_s - E(X_s))\right]$$

$$= E\left[\left((t\mu + \varepsilon_t + \dots + \varepsilon_1) - t\mu\right)\left((s\mu + \varepsilon_s + \dots + \varepsilon_1) - s\mu\right)\right]$$

$$= E\left[(\varepsilon_t + \varepsilon_{t-1} + \dots \varepsilon_1)(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1)\right]$$



$$= E\left[\underbrace{(\varepsilon_t + \varepsilon_{t-1} + \dots + \varepsilon_1)}(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1)\right] =$$

we can write as

$$(\varepsilon_t + \varepsilon_{t-1} + \dots + \varepsilon_{s+1}) + (\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1)$$

$$= E\left[\left\{(\varepsilon_t + \varepsilon_{t-1} + \dots + \varepsilon_{s+1}) + (\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1)\right\} \cdot\right.$$

$$\left.\cdot (\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1)\right] =$$

$$= E\left[(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1)^2 + (\varepsilon_t \varepsilon_s + \varepsilon_t \varepsilon_{s-1} + \dots + \varepsilon_{t-1}\varepsilon_s + \dots)\right]$$

$$= \underbrace{E\left[(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1)^2\right]} + \underbrace{E(\varepsilon_t \varepsilon_s + \varepsilon_t \varepsilon_{s-1} + \dots)} =$$

$$= E(\varepsilon_s^2) + E(\varepsilon_{s-1}^2) + \dots$$

$= \text{VAR OF } \varepsilon_s = \sigma_\varepsilon^2$ $\quad$ $= \sigma_\varepsilon^2$

$$= \underbrace{E(\varepsilon_t \varepsilon_s)}_{=0} + \underbrace{E(\varepsilon_t \varepsilon_{s-1})}_{=0} + \dots =$$

BECAUSE ε IS WHITE NOISE WITH ZERO COVARIANCE

$$= \underbrace{\sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \dots +}_{= s\sigma_\varepsilon^2} 0 + 0 + 0 + \dots$$

$$= s\sigma_\varepsilon^2 \longrightarrow \operatorname{Cov}(X_t, X_s) = s\sigma_\varepsilon^2 = \min(t,s)\,\sigma_\varepsilon^2$$

Problem 7.2

$$X_t = X_{t-1} + \varepsilon_t \qquad \text{where } \varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$$

THEN:

$$X_{t+1} = X_t + \varepsilon_{t+1}$$

$$X_{t+2} = X_{t+1} + \varepsilon_{t+2}$$

$$\vdots$$

$$X_{t+K} = X_{t+K-1} + \varepsilon_{t+K}$$

i) $\hat{X}_t(1) = ?$

$$\hat{X}_t(1) = E[X_{t+1} | F_t] = [X_{t+1}]$$

$$\hat{X}_t(1) = [X_t + \varepsilon_{t+1}]$$

$$= \underbrace{[X_t]}_{= X_t} + \underbrace{[\varepsilon_{t+1}]}_{= 0}$$

$\cancel{E(\varepsilon_t}$ $E(\varepsilon_{t+1} | F_t) = 0$

BECAUSE PAST VALUES CANNOT PREDICT FUTURE ERROR TERMS

$$\hat{X}_t(1) = X_t$$

ii) $\hat{X}_t(2) = ?$

$$\hat{X}_t(2) = [X_{t+2}]$$

$$= [X_{t+1} + \varepsilon_{t+2}]$$

$$= \underbrace{[X_{t+1}]}_{\downarrow} + \underbrace{[\varepsilon_{t+2}]}_{=0}$$

$$\hat{X}_t(1) = X_t$$

$\rightarrow$ $\hat{X}_t(2) = X_t$

$\rightarrow$ $\hat{X}_t(3) = [X_{t+3}]$

$$= \underbrace{[X_{t+2}]}_{\hat{X}_t(2) = X_t} + \underbrace{[\varepsilon_{t+3}]}_{=0}$$

$$= X_t$$

iii) $\hat{X}_t(K)$?

$$\hat{X}_t(K) = [X_{t+K}]$$

$$= \underbrace{[X_{t+K-1}]}_{\downarrow} + \underbrace{[\varepsilon_{t+K}]}_{=0}$$

$$\downarrow$$

$$\hat{X}_t(k-1) = X_t$$

$$\rightarrow \hat{X}_t(k) = X_t$$

THE SERIES IS <u>NOT</u> MEAN REVERTING

$$E(X_t) = E(X_{t-1} + \varepsilon_t)$$

$$\underbrace{E(X_t)}_{\mu} = \underbrace{E(X_{t-1})}_{\downarrow} + \underbrace{E(\varepsilon_t)}_{=0}$$

$$E(X_{t-2} + \varepsilon_{t-1})$$

$$\vdots$$

$$E(X_0) = 0$$

$$\mu = 0$$

Problem 7.3

$$X_t = X_{t-1} + Z_t$$

where $Z_t = 3 + 0.5\, Z_{t-1} + 0.1\, Z_{t-2} + \varepsilon_t$

$$\varepsilon_t \sim WN(0, \sigma^2_\varepsilon)$$

$$\hat{X}_t(1) = ?$$

- FIRST WE DEFINE X_{t+1}

$$X_{t+1} = X_t + Z_{t+1}$$

- LET'S FIND Z_{t+1} AND 1. STEP AHEAD FORECAST $\hat{Z}_t(1)$

$$Z_{t+1} = 3 + 0.5\, Z_t + 0.1\, Z_{t-1} + \varepsilon_t$$

$$\hat{Z}_t(1) = E[Z_{t+1} | F_t] = [Z_{t+1}]$$

$$= [3 + 0.5\, Z_t + 0.1\, Z_{t-1} + \varepsilon_t]$$

$$= 3 + 0.5\, \underbrace{[Z_t]}_{Z_t} + 0.1\, \underbrace{[Z_{t-1}]}_{= Z_{t-1}} + \underbrace{[\varepsilon_t]}_{= 0}$$

$$= 3 + 0.5\, Z_t + 0.1\, Z_{t-1}$$

WE KNOW THAT

$$X_t = X_{t-1} + Z_t \iff Z_t = X_t - X_{t-1}$$

THEREFORE

$$3 + 0.5\,Z_t + 0.1\,Z_{t-1} =$$

$$= 3 + 0.5\,(X_t - X_{t-1}) + 0.1\,(X_{t-1} - X_{t-2})$$

$$= 3 + 0.5\,X_t - 0.4\,X_{t-1} - 0.1\,X_{t-2}$$

- FIND $\hat{X}_t(1)$

$$\hat{X}_t(1) = E[X_{t+1} \mid F_t] = [X_{t+1}]$$

$$= [X_t + Z_{t+1}] =$$

$$= [X_t] + [Z_{t+1}]$$

$$= X_t + \hat{Z}_t(1)$$

$$= X_t + (3 + 0.5X_t - 0.4X_{t-1} - 0.1X_{t-2})$$

$$= 3 + 1.5X_t - 0.4X_{t-1} - 0.1X_{t-2}$$