ECOM073: Topics in Financial Econometrics

Exercise 7.

Problem 7.1. Assume that X_t is a random walk with a drift:

$$X_t = \mu + X_{t-1} + \varepsilon_t, \quad t = 1, 2, ...$$

with the initial value $X_0 = 0$, where ε_t is a white noises with zero mean and variance σ_ε^2.

Show that

(i) $E[X_t] = \mu t$.
(ii) $Var(X_t) = t\sigma_\varepsilon^2$.
(iii) $Cov(X_t, X_s) = \min(t, s)\sigma_\varepsilon^2$.

Solution 1. (i) We can write

$$\begin{aligned}
X_t &= \mu + X_{t-1} + \varepsilon_t = \mu + (\mu + X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= 2\mu + X_{t-2} + \varepsilon_{t-1} + \varepsilon_t = 3\mu + X_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\
&= .... = t\mu + X_0 + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + ... + \varepsilon_1 \\
&= t\mu + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + ... + \varepsilon_1,
\end{aligned}$$

since $X_0 = 0$. Then

$$\begin{aligned}
E[X_t] &= E[t\mu + \varepsilon_t + \varepsilon_{t-1} + ... + \varepsilon_1] \\
&= t\mu + E[\varepsilon_t] + E[\varepsilon_{t-1}] + ... + E[\varepsilon_1] = t\mu + 0 + 0 + ... + 0 = t\mu.
\end{aligned}$$

(ii)

$$\begin{aligned}
Var(X_t) &= E[(X_t - E[X_t])^2] = E[X_t^2] \\
&= E[(\varepsilon_t + \varepsilon_{t-1} + ... + \varepsilon_1)^2] = E[\varepsilon_t^2] + E[\varepsilon_{t-1}^2] + ... + E[\varepsilon_1^2] \\
&= \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + ... + \sigma_\varepsilon^2 = t\sigma_\varepsilon^2.
\end{aligned}$$

because ε_t is a white noise and therefore $E[\varepsilon_i \varepsilon_j] = 0$, if $i \neq j$; and $E[\varepsilon_i \varepsilon_j] = \sigma_\varepsilon^2$, if $i = j$.

(iii) By definition,

$$Cov(X_t, X_s) = E[(X_t - E[X_t])(X_s - E[X_s])].$$

Since $E[X_t] = t\mu$ and

$$X_t = t\mu + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + ... + \varepsilon_1,$$

then for $t \geq s$,

$$\begin{aligned} Cov(X_t, X_s) &= E[X_t X_s] = E[(\varepsilon_t + \varepsilon_{s-1} + ... + \varepsilon_1)(\varepsilon_s + \varepsilon_{s-1} + ... + \varepsilon_1)] \\ &= E[(\{\varepsilon_s + \varepsilon_{s-1} + ... + \varepsilon_1\} + \{\varepsilon_{s+1} + ... + \varepsilon_t\})(\varepsilon_s + \varepsilon_{s-1} + ... + \varepsilon_1)] \\ &= E[(\varepsilon_s + \varepsilon_{t-1} + ... + \varepsilon_1)^2] + E[(\varepsilon_{s+1} + ... + \varepsilon_t)(\varepsilon_s + \varepsilon_{s-1} + ... + \varepsilon_1)] \\ &= s\sigma_\varepsilon^2 + 0 = s\sigma_\varepsilon^2. \end{aligned}$$

Therefore

$$Cov(X_t, X_s) = s\sigma_\varepsilon^2 = \min(t, s)\sigma_\varepsilon^2.$$

Problem 7.2.
(a) Suppose that X_t follows a unit root model

$$X_t = X_{t-1} + \varepsilon_t$$

with initial value $X_0 = 0$ where ε_t as a white noise process with zero mean and variance σ_ε^2

(i) Obtain the 1-step ahead prediction $X_t(1)$ at the origin t.
(ii) Derive 2-step ahead prediction $X_t(1)$
(iii) Is the forecast mean reverting?

Solution. (i) We have that

$$\hat{X}_t(1) = E[X_{t+1}|F_t] = [X_{t+1}] = [X_t + \varepsilon_{t+1}] = [X_t] + [\varepsilon_{t+1}] = X_t + 0 = X_t.$$

(ii) Similarly,

$$\hat{X}_t(2) = E[X_{t+2}|F_t] = [X_{t+2}] = [X_{t+1} + \varepsilon_{t+2}] = [X_{t+1}] + [\varepsilon_{t+2}] = \hat{X}_t(1) = X_t.$$

(iii) We find that

$$\hat{X}_t(k) = [X_{t+k}] = [X_{t+k-1} + \varepsilon_{t+k}] = [X_{t+k-1}]$$

$$= \textit{continuing as above} = [X_{t+k-2}] = ... = [X_t] = X_t.$$

We showed in the first exercise that $EX_0 = 0$, which is different from X_t. So the forecast is not mean reverting.

Problem 7.3.
(a) Suppose that X_t follows a unit root model and

$$X_t = X_{t-1} + z_t$$

where z_t is an AR(2) time series:

$$z_t = 3 + 0.5z_{t-1} + 0.1z_{t-2} + \varepsilon_t,$$

where ε_t is a white noise with the mean 0 and variance σ_ε^2.
Find the 1-step ahead forecast $X_t(1)$.

Solution. First we find the 1-step ahead forecast $z_t(1)$ of z_{t+1}:

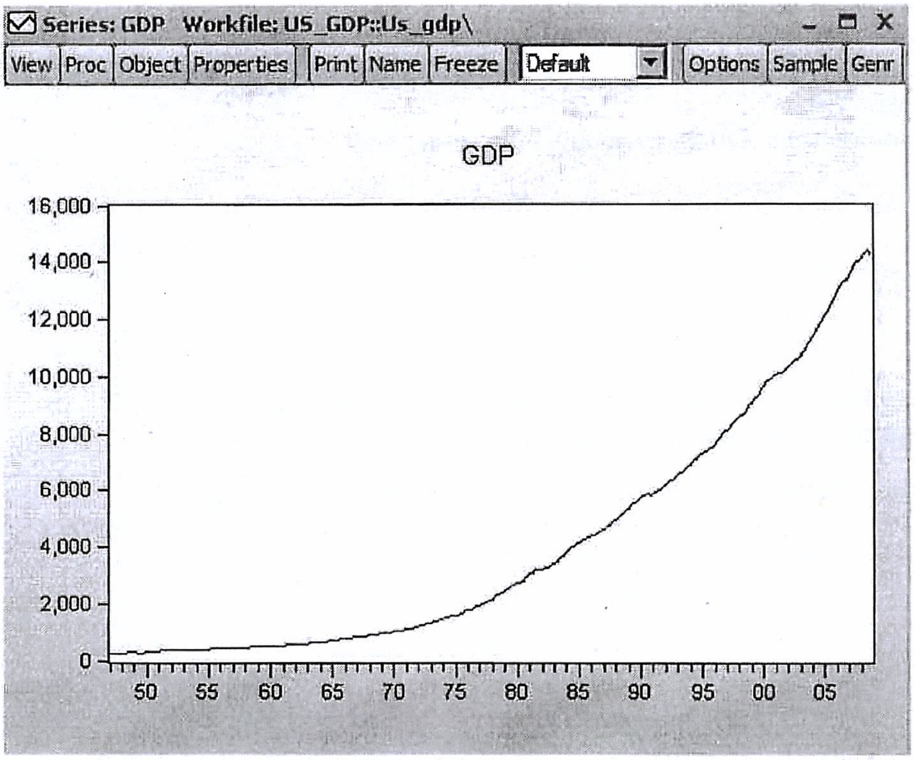
$$\begin{aligned}
\hat{z}_t(1) &= E[z_{t+1}|F_t] = [z_{t+1}] \\
&= [3 + 0.5z_t + 0.1z_{t-1} + \varepsilon_{t+1}] \\
&= 3 + 0.5[z_t] + 0.1[z_{t-1}] + [\varepsilon_{t+1}] \\
&= 3 + 0.5z_t + 0.1z_{t-1} \\
&= 3 + 0.5(X_t - X_{t-1}) + 0.1(X_{t-1} - X_{t-2}).
\end{aligned}$$

Then

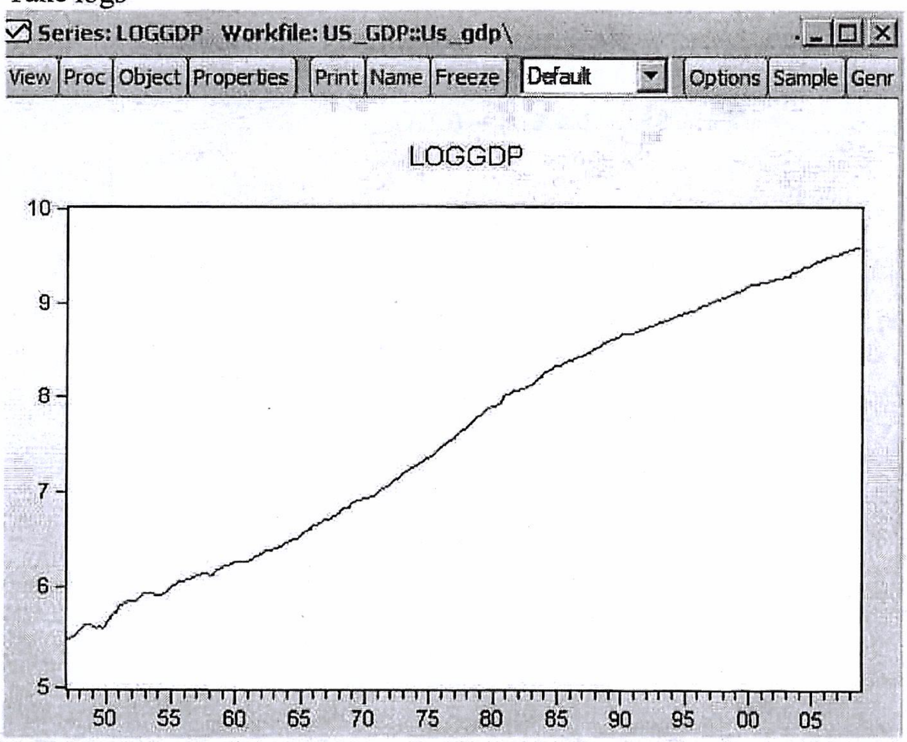
$$\begin{aligned}
\hat{X}_t(1) &= [X_{t+1}] = [X_t + z_{t+1}] = [X_t] + [z_{t+1}] = X_t + \hat{z}_t(1) \\
&= X_t + 3 + 0.5(X_t - X_{t-1}) + 0.1(X_{t-1} - X_{t-2}) \\
&= 3 + 1.5X_t - 0.4X_{t-1} - 0.1X_{t-2}.
\end{aligned}$$

Problem 7.4. (EViews exercise) Test for unit root in the U.S. quarterly GDP data. Fit a model to that data.

Solution. 1 To test for unit root, use the $\log X_t$, $t = 1, \cdots, 243$ observations shown in the graph LOGGDP.



Take logs



2. To test for unit root use the Augmented D-F test with automatic selection of the number of lags p (by AIC criterion). It gives $p = 4$.

$p = 0.9956$ value shows that we do not reject unit root.

Output shows that to the data $Y_t = \log X_t$ we can fit the model (with $\Delta Y_t = Y_t - Y_{t-1}$):

$$\Delta Y_t = c + \text{"}trend\text{"}t + \beta_c Y_{t-1} + \delta_1 \Delta Y_{t-1} + ... + \delta_4 \Delta Y_{t-4} + \varepsilon_t.$$

Estimation shows that $\beta_c = -0.000214$, $c = 0.011618$ and "trend"$= -6.83E-05$ are not significant, as w ll as the coefficients $\delta_3 = 0.0704$ and $\delta_4 = 0.0658$. The fitted model is

$$\Delta Y_t = 0.4221 \Delta Y_{t-1} + 0.1941 \Delta Y_{t-2} + \varepsilon_t.$$

USE LGDP (log of GDP)

Estimation and ADF test on log(GDP)

EViews - [Series: LGDP Workfile: US_GDP::Us_gdp\]

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Augmented Dickey-Fuller Unit Root Test on LGDP

Null Hypothesis: LGDP has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 4 (Fixed)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-0.039245	0.9956
Test critical values:	1% level	-3.996271	
	5% level	-3.428426	
	10% level	-3.137619	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(LGDP)
Method: Least Squares
Date: 03/06/12 Time: 18:18
Sample (adjusted): 1948Q2 2008Q4
Included observations: 243 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
LGDP(-1)	-0.000214	0.005462	-0.039245	0.9687
D(LGDP(-1))	0.422150	0.066154	6.381327	0.0000
D(LGDP(-2))	0.194117	0.070375	2.758317	0.0063
D(LGDP(-3))	-0.132396	0.070432	-1.879774	0.0614
D(LGDP(-4))	-0.042067	0.065829	-0.639041	0.5234
C	0.011618	0.029132	0.398790	0.6904
@TREND(1947Q1)	-6.83E-06	9.84E-05	-0.069416	0.9447

R-squared	0.253101	Mean dependent var	0.016456
Adjusted R-squared	0.234113	S.D. dependent var	0.011168
S.E. of regression	0.009773	Akaike info criterion	-6.389901
Sum squared resid	0.022543	Schwarz criterion	-6.289278
Log likelihood	783.3730	Hannan-Quinn criter.	-6.349371
F-statistic	13.32888	Durbin-Watson stat	1.977094
Prob(F-statistic)	0.000000		

We cannot reject the null of the presence of a unit root.

3. The above results suggest that $z_t = Y_t - Y_{t-1}$ follows AR(2) model. If we test for the unit root in z_t, we find no unit root.

Take differences and estimate (this is a process with a constant and a trend).

genr diflgdp = d(loggdp)

EViews - [Series: DIFLGDP Workfile: US_GDP::Us_gdp\]
File Edit Object View Proc Quick Options Add-ins Window Help
View Proc Object Properties Print Name Freeze Sample Genr Sheet Graph Stats Ident
Augmented Dickey-Fuller Unit Root Test on DIFLGDP

Null Hypothesis: DIFLGDP has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 4 (Fixed)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-6.948482	0.0000
Test critical values:	1% level	-3.996431	
	5% level	-3.428503	
	10% level	-3.137665	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(DIFLGDP)
Method: Least Squares
Date: 03/06/12 Time: 18:23
Sample (adjusted): 1948Q3 2008Q4
Included observations: 242 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
DIFLGDP(-1)	-0.625886	0.090075	-6.948482	0.0000
D(DIFLGDP(-1))	0.042208	0.086496	0.487980	0.6260
D(DIFLGDP(-2))	0.219443	0.079212	2.770327	0.0060
D(DIFLGDP(-3))	0.109443	0.075679	1.446146	0.1495
D(DIFLGDP(-4))	0.114535	0.065271	1.754753	0.0806
C	0.011686	0.002112	5.531876	0.0000
@TREND(1947Q1)	-1.14E-05	9.05E-06	-1.256593	0.2101

R-squared	0.297334	Mean dependent var	-0.000169
Adjusted R-squared	0.279393	S.D. dependent var	0.011463
S.E. of regression	0.009730	Akaike info criterion	-6.398606
Sum squared resid	0.022250	Schwarz criterion	-6.297687
Log likelihood	781.2314	Hannan-Quinn criter.	-6.357952
F-statistic	16.57341	Durbin-Watson stat	1.946147
Prob(F-statistic)	0.000000		

No unit root found, as expected.