Topic: UNIT ROOT NONSTATIONARITY

CONSIDER AR(1) MODEL:

$$X_t = \phi X_{t-1} + \varepsilon_t$$

- IF $|\phi| < 1$ then THE MODEL HAS A STATIONARY SOLUTION
- IF $\phi = 1$ then THE MODEL IS A UNIT ROOT MODEL WHICH IS NON-STATIONARY

SUCH MODEL DOES NOT SATISFY CONDITION OF COVARIANCE STATIONARITY OF AN AR(1) MODEL

A time series X_t is covariance stationary if:

$$E(X_t) = \mu_x$$

$$Var(X_t) = \sigma_x^2$$

$$Cov(X_t, X_{t-k}) = \gamma_k$$

IN THIS CLASS WE ARE GOING TO CONSIDER THE RANDOM WALK AND THE RANDOM WALK WITH DRIFT MODELS.

- A TIME SERIES X_t IS CALLED A RANDOM WALK IF

$$X_t = X_{t-1} + \varepsilon_t \qquad t = 0, 1, \dots,$$

where ε_t IS A ZERO MEAN WHITE NOISE

- A RANDOM WALK WITH A DRIFT CAN BE WRITTEN AS:

$$X_t = \mu + X_{t-1} + \varepsilon_t$$

→ A RANDOM-WALK IS COVARIANCE NON-STATIONARY MODEL

Problem 7.1

$$X_t = \mu + X_{t-1} + \varepsilon_t \qquad t = 1, 2, \ldots$$

where $X_0 = 0$

ε_t IS $WN(0, \sigma_\varepsilon^2)$

X_t CAN BE REWRITTEN AS:

$$X_t = \mu + X_{t-1} + \varepsilon_t$$

$$= \mu + (\mu + X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$

$$= 2\mu + X_{t-2} + \varepsilon_{t-1} + \varepsilon_t$$

$$= 2\mu + (\mu + X_{t-3} + \varepsilon_{t-2}) + \varepsilon_{t-1} + \varepsilon_t$$

$$= 3\mu + X_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t$$

$$= \ldots\ldots$$

$$= t\mu + \underbrace{X_0}_{=0} + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots + \varepsilon_1$$

we assumed $X_0 = 0$

ε_1 IS COMING FROM $\varepsilon_{t-(t-1)}$

$$= t\mu + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots + \varepsilon_1$$

(i) $E(X_t) = t\mu$

$$E(X_t) = E(t\mu + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \dots + \varepsilon_1)$$

$$= \underbrace{E(t\mu)}_{=t\mu} + \underbrace{E(\varepsilon_t)}_{=0} + \underbrace{E(\varepsilon_{t-1})}_{=0} + \dots + \underbrace{E(\varepsilon_1)}_{=0}$$

ε_t IS A WHITE NOISES SEQUENCE WITH ZERO MEAN

SO

$$E(X_t) = t\mu$$

(ii) $Var(X_t) = t\,\sigma_\varepsilon^2$

$$Var(X_t) = E[(X_t - E(X_t))^2] =$$

$$= E[((t\mu + \varepsilon_t + \varepsilon_{t-1} + \dots + \varepsilon_1) - t\mu)^2]$$

$$= E[(\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \dots + \varepsilon_1)^2]$$

$$= \underbrace{E[\varepsilon_t^2]}_{=\sigma_\varepsilon^2} + \underbrace{E[\varepsilon_{t-1}^2]}_{=\sigma_\varepsilon^2} + \dots + \underbrace{E[\varepsilon_1^2]}_{=\sigma_\varepsilon^2}$$

$$Var(\varepsilon_t) = E[(\varepsilon_t - \underbrace{E(\varepsilon_t)}_{=0})^2] = E[\varepsilon_t^2]$$

$$= \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \dots + \sigma_\varepsilon^2$$

$$= t\,\sigma_\varepsilon^2$$

(iii) $Cov(X_t, X_s) = \min(t, s)\sigma_\varepsilon^2$

WE KNOW $X_t = \mu + X_{t-1} + \varepsilon_t$

$$X_t = t\mu + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \dots + \varepsilon_1$$

AND $E(X_t) = t\mu$

WE CAN NOW DEFINE X_s AS

$$X_s = s\mu + \varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1$$

where $t \geq s$

WE OBTAIN THAT $E(X_s) = s\mu$

- BY THE DEFINITION OF COVARIANCE:

$$Cov(X_t, X_s) = E\left[(X_t - E(X_t))(X_s - E(X_s))\right] =$$

$$= E\left[\left((t\mu + \varepsilon_t + \varepsilon_{t-1} + \dots + \varepsilon_1) - t\mu\right) \cdot \right.$$

$$\left. \cdot \left((s\mu + \varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1) - s\mu\right)\right] =$$

$$= E\left[(\varepsilon_t + \varepsilon_{t-1} + \dots + \varepsilon_1)(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1)\right] =$$

IT CAN BE WRITTEN AS:

$$\left(\{\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1\} + \{\varepsilon_{s+1} + \varepsilon_{s+2} + \dots + \varepsilon_t\}\right)$$

$$= E\Big[\big(\{\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1\} + \{\varepsilon_{s+1} + \dots + \varepsilon_t\}\big) \cdot$$

$$\cdot \big(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1\big)\Big] =$$

$$= E\Big[\big(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1\big)^2 + \big(\varepsilon_{s+1} + \dots + \varepsilon_t\big)\big(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1\big)\Big] =$$

$$= \underbrace{E\Big[\big(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1\big)^2\Big]}_{= s\sigma_\varepsilon^2} + \underbrace{E\Big[\big(\varepsilon_{s+1} + \dots + \varepsilon_t\big)\big(\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1\big)\Big]}_{= 0} =$$

BECAUSE

$$E(\varepsilon_s^2) + E(\varepsilon_{s-1}^2) + \dots + E(\varepsilon_1^2) =$$

$$= \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \dots + \sigma_\varepsilon^2 =$$

$$= s\sigma_\varepsilon^2$$

BECAUSE

$$E\big[\varepsilon_{s+1}\varepsilon_s + \varepsilon_{s+1}\varepsilon_{s-1} + \dots + \varepsilon_t\varepsilon_1\big] =$$

$$= E\big[\varepsilon_{s+1}\varepsilon_s\big] + E\big[\varepsilon_{s+1}\varepsilon_{s-1}\big] + \dots =$$

$$= 0 + 0 + \dots = 0$$

ε IS A WHITE NOISE SEQUENCE AND THEREFORE

$$E(\varepsilon_i \varepsilon_j) = 0 \quad \text{if } i \neq j$$

$$= s\sigma_\varepsilon^2 + 0 =$$

$$= s\sigma_\varepsilon^2$$

WE CAN CONCLUDE:

$$\mathrm{Cov}(X_t, X_s) = s\sigma_\varepsilon^2 = \min(t, s)\,\sigma_\varepsilon^2$$

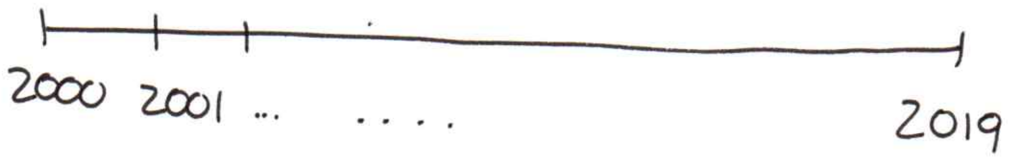
Additional Notes:

$\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \dots + \varepsilon_1$ CAN BE WRITTEN AS

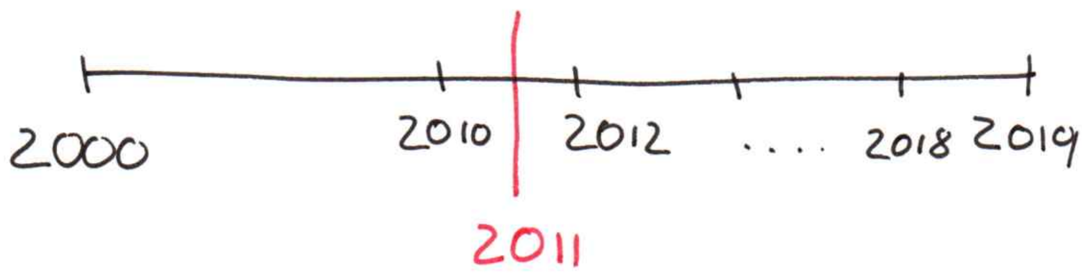
$$\{\varepsilon_s + \varepsilon_{s-1} + \dots + \varepsilon_1\} + \{\varepsilon_{s+1} + \varepsilon_{s+2} + \dots + \varepsilon_{t-1} + \varepsilon_t\}$$

WE CAN EXPLAIN IT USING AN EXAMPLE:

WE CONSIDER THE TIME PERIOD 2000 - 2019



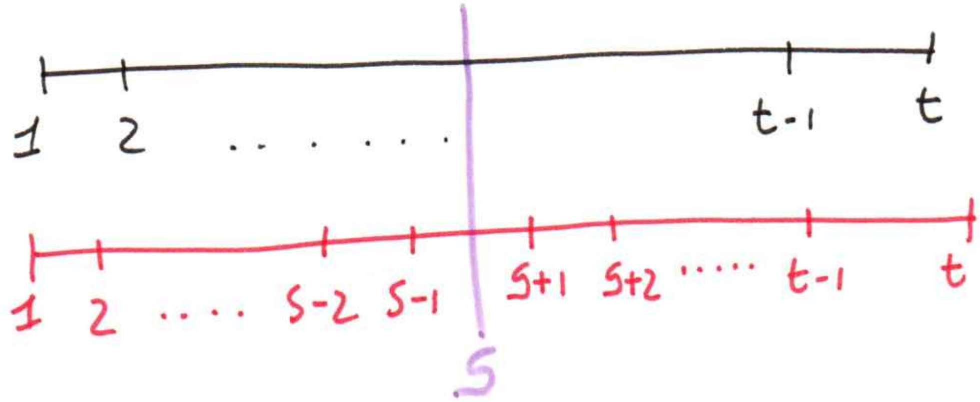
AND WE WANT TO TEST THE CORRELATION BETWEEN X_{2019} AND X_{2011}



the sample could be split in two

WE CAN GENERALIZED IT BY USING t

AND WE WANT TO COMPUTE THE CORRELATION BETWEEN X_t AND X_s where $t > s$



SO THE PERIOD $1 - t$ CAN BE DIVIDED INTO TWO PERIODS:

$(1 - s)$ AND $(s+1 - t)$

THE ABOVE EXAMPLE EXPLAINS WHY

$$\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \dots + \varepsilon_1 =$$

$$= (\varepsilon_1 + \dots + \varepsilon_{s-1} + \varepsilon_s) + (\varepsilon_{s+1} + \dots + \varepsilon_{t-1} + \varepsilon_t)$$

Problem 7.2

CONSIDER $X_t = X_{t-1} + \varepsilon_t$

WITH INITIAL VALUE $X_0 = 0$ AND

ε_t AS A WHITE NOISE WITH ZERO MEAN AND VARIANCE σ^2_ε

WE KNOW $X_t = X_{t-1} + \varepsilon_t$

THEN $X_{t+1} = X_t + \varepsilon_{t+1}$

AND $X_{t+2} = X_{t+1} + \varepsilon_{t+2}$

(i) OBTAIN THE 1-STEP AHEAD PREDICTION $\hat{X}_t(1)$

THE 1-STEP AHEAD FORECAST IS DEFINED BY:

$$\hat{X}_t(1) = E\left[X_{t+1} \mid F_t\right] = \left[X_{t+1}\right]$$

WE HAVE THAT:

$$\hat{X}_t(1) = E\left[X_{t+1} \mid F_t\right] = \left[X_{t+1}\right] =$$

$$= \left[X_t + \varepsilon_{t+1}\right] =$$

$$= \underbrace{\left[X_t\right]}_{=X_t} + \underbrace{\left[\varepsilon_{t+1}\right]}_{=0} = X_t$$

BECAUSE X_t IS KNOWN WHEN WE KNOW F_t

BECAUSE ε_{t+1} IS INDEPENDENT OF THE "HISTORY" F_t

(ii) DERIVE 2-STEP AHEAD PREDICTION $\hat{X}_t(2)$

$$\hat{X}_t(2) = E[X_{t+2} \mid F_t] = [X_{t+2}] =$$

$$= [X_{t+1} + \varepsilon_{t+2}] =$$

$$= \underbrace{[X_{t+1}]}_{= \hat{X}_t(1) = X_t} + \underbrace{[\varepsilon_{t+2}]}_{=0}$$

SO $\hat{X}_t(2) = X_t + 0 = X_t$

THE FORECAST OF A RANDOM-WALK MODEL IS THE LAST OBSERVATION X_t

FOR ANY K-STEP AHEAD FORECAST

$$\hat{X}_t(K) = X_t$$

THEREFORE:

FORECASTING IS <u>NOT</u> MEAN REVERTING

(iii) IS THE FORECAST MEAN REVERTING?

NO, THE FORECAST IS NOT MEAN REVERTING BECAUSE $E(X_t) = 0$ WHICH IS DIFFERENT FROM X_t

WE KNOW $X_t = X_{t-1} + \varepsilon_t$

$$X_t = (X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$

$$X_t = X_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t$$

$$\vdots$$

$$X_t = \underbrace{X_0}_{=0} + \varepsilon_1 + \dots + \varepsilon_{t-1} + \varepsilon_t$$

BECAUSE INITIAL VALUE $X_0 = 0$

$$E(X_t) = E(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_t)$$

$$= E(\varepsilon_1) + \dots + E(\varepsilon_t) =$$

$$= 0$$

THIS IS DIFFERENT FROM:

$$\hat{X}_t(1) = X_t$$

$$\hat{X}_t(2) = X_t$$

$$\vdots$$

$$\hat{X}_t(k) = X_t$$

Problem 7.3

CONSIDER $X_t = X_{t-1} + Z_t$

where Z_t IS AN AR(2) MODEL:

$$Z_t = 3 + 0.5\, Z_{t-1} + 0.1\, Z_{t-2} + \varepsilon_t$$

where ε_t IS WN $(0, \sigma^2_\varepsilon)$

- FIND 1-STEP AHEAD FORECAST $\hat{X}_t(1)$

- FIRST, WE NEED TO FIND 1-STEP AHEAD FORECAST OF Z_{t+1}

$$\hat{Z}_t(1) = E[Z_{t+1} \mid F_t] = [Z_{t+1}]$$

$$= [3 + 0.5\, Z_t + 0.1\, Z_{t-1} + \varepsilon_{t+1}]$$

$$= 3 + 0.5 \underbrace{[Z_t]}_{= Z_t} + 0.1 \underbrace{[Z_{t-1}]}_{= Z_{t-1}} + \underbrace{[\varepsilon_{t+1}]}_{= 0} =$$

Z_t AND Z_{t-1} ARE KNOWN WHEN WE KNOW F_t

BECAUSE ε_{t+1} IS INDEPENDENT OF THE HISTORY OF F_t

$$= 3 + 0.5\, Z_t + 0.1\, Z_{t-1} =$$

$$= 3 + 0.5\,(X_t - X_{t-1}) + 0.1\,(X_{t-1} - X_{t-2})$$

we know $X_t = X_{t-1} + Z_t$ THEN $Z_t = X_t - X_{t-1}$

- THEN

$$\hat{X}_t(1) = E[X_{t+1} | F_t] = [X_{t+1}]$$

$$= [X_t + Z_{t+1}] =$$

$$= \underbrace{[X_t]}_{= X_t} + \underbrace{[Z_{t+1}]}_{= \hat{Z}_t(1)} =$$

$$= 3 + 0.5(X_t - X_{t-1}) + 0.1(X_{t-1} - X_{t-2})$$

$$= X_t + 3 + 0.5(X_t - X_{t-1}) + 0.1(X_{t-1} - X_{t-2}) =$$

$$= X_t + 3 + 0.5X_t - 0.5X_{t-1} + 0.1X_{t-1} - 0.1X_{t-2} =$$

$$= 3 + 1.5X_t - 0.4X_{t-1} - 0.1X_{t-2}$$

Problem 7.4

STEPS TO FOLLOW:

- PLOT GDP SERIES
- GENERATE LOG(GDP) AND PLOT THE NEW SERIES
- TEST FOR UNIT ROOT

H_0: SERIES CONTAINS UNIT ROOT

H_1: SERIES IS STATIONARY

SINCE P_VALUE $> \alpha$ THEN

WE FAIL TO REJECT H_0

$\rightarrow$ LOGGDP HAS AT LEAST A UNIT ROOT

- OUTPUT SHOWS WE CAN FIT THE MODEL:

$$\Delta Y_t = c + \text{"trend"}\, t + \beta_c Y_{t-1} + \delta_1 \Delta Y_{t-1} + \delta_2 \Delta Y_{t-2} + \varepsilon_t$$

where $\Delta Y_t = Y_t - Y_{t-1}$

AND Y_t IS LOGGDP

OUTPUT SUGGESTS THAT:

$$\Delta Y_t = 0.4221\, \Delta Y_{t-1} + 0.1941\, \Delta Y_{t-2} + \varepsilon_t$$

THE COEFFICIENTS c, trend, β_c, δ_3, δ_4
ARE NOT SIGNIFICANT

- THE SERIES IS NOT STATIONARY SO
  WE TEST THE FIRST DIFFERENCE

$$LOGGDP_t - LOGGDP_{t-1}$$

THE FIRST DIFFERENCE IS STATIONARY
→ P-VALUE $< \alpha$ THEN WE REJECT H_0