Exercise 6.1

AR(1) $\quad X_t = \phi X_{t-1} + \varepsilon_t \quad$ where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

BY THE ASSUMPTION OF PARAMETER STABILITY

$$X_{t+1} = \phi X_t + \varepsilon_{t+1}$$

$$X_{t+2} = \phi X_{t+1} + \varepsilon_{t+2}$$

$$X_{t+3} = \phi X_{t+2} + \varepsilon_{t+3}$$

$$\vdots$$

$$X_{t+K} = \phi X_{t+K-1} + \varepsilon_{t+K}$$

1) 1- STEP AHEAD FORECAST

$$\hat{X}_t(1) = E[X_{t+1} | F_t] = [X_{t+1}]$$

$$\hat{X}_t(1) = ?$$

$$\hat{X}_t(1) = [X_{t+1}]$$

$$= [\phi X_t + \varepsilon_{t+1}]$$

$$= [\phi X_t] + [\varepsilon_{t+1}] =$$

$$= \phi\,[X_t] + [\varepsilon_{t+1}]$$

$$[X_t] = E[X_t \mid F_t] = X_t$$

GIVEN THE INFORMATION AT TIME t, WE KNOW X_t

$$[\varepsilon_{t+1}] = E[\varepsilon_{t+1} \mid F_t] = 0$$

BECAUSE

- ε_t IS A WHITE NOISE
- FUTURE IS NOT CORR. WITH THE PAST

$$= \phi X_t + 0$$

SO $\hat{X}_t(1) = \phi X_t$

→ FORECAST ERROR

$$e_t(1) = X_{t+1} - \hat{X}_t(1)$$

$$= (\cancel{\phi X_t} + \varepsilon_{t+1}) - (\cancel{\phi X_t})$$

$$= \varepsilon_{t+1}$$

$$Var(e_t(1)) = Var(\varepsilon_{t+1}) = \sigma^2_\varepsilon$$

ε_t IS A WHITE NOISE SO VARIANCE IS CONSTANT

$Var\ \varepsilon_t = Var\ \varepsilon_{t-1} = Var\ \varepsilon_{t+1}$

$$= \phi \varepsilon_{t+1} + \varepsilon_{t+2}$$

$$\rightarrow Var(e_t(2)) = \cancel{X\varepsilon_{t+2}}\, Var(\phi \varepsilon_{t+1} + \varepsilon_{t+2}) =$$

$$\boxed{\text{VAR. RULE: } Var(aX + Y) = a^2 Var(X) + Var(Y) + 2a \underbrace{Cov(X, Y)}_{=0}}$$

if x is not corr with y

$$= \phi^2 \underbrace{Var(\varepsilon_{t+1})}_{\sigma_\varepsilon^2} + \underbrace{Var(\varepsilon_{t+2})}_{\sigma_\varepsilon^2}$$

$$= \phi^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^2$$

$$= (\phi^2 + 1)\sigma_\varepsilon^2$$

$$Var(e_t(2)) > Var(e_t(1))$$

$$(\phi^2 + 1)\sigma_\varepsilon^2 > \sigma_\varepsilon^2$$

2) 2-STEP AHEAD FORECAST

$$\hat{X}_t(2) = ?$$

$$\hat{X}_t(2) = E[X_{t+2} | F_t] = [X_{t+2}]$$

$$= [\phi X_{t+1} + \varepsilon_{t+2}]$$

$$= \phi [X_{t+1}] + [\varepsilon_{t+2}] =$$

↳ $E[\varepsilon_{t+2} | F_t] = 0$

1-step ahead forecast

$$[X_{t+1}] = \hat{X}_t(1)$$

$$= \phi X_t$$

$$= \phi \underbrace{\hat{X}_t(1)} + 0$$

$$= \phi (\phi X_t) =$$

$$= \phi^2 X_t$$

→ FORECAST ERROR

$$e_t(2) = X_{t+2} - \hat{X}_t(2)$$

$$= (\phi X_{t+1} + \varepsilon_{t+2}) - \phi \hat{X}_t(1)$$

$$= \phi \underbrace{(X_{t+1} - \hat{X}_t(1))} + \varepsilon_{t+2}$$

↳ $\hat{e}_t(1) = \varepsilon_{t+1}$

3) $\hat{X}_t(K) = ?$

$$\hat{X}_t(K) = [X_{t+K}]$$

$$= [\phi X_{t+K-1} + \varepsilon_{t+K}]$$

$$= \phi \underbrace{[X_{t+K-1}]}_{= \hat{X}_t(K-1)} + \underbrace{[\varepsilon_{t+K}]}_{= 0}$$

$$= \phi \underbrace{\hat{X}_t(K-1)}_{= \phi \hat{X}_t(K-2)} + 0$$

$$= \phi^2 \underbrace{\hat{X}_t(K-2)}_{= \phi \hat{X}_t(K-3)}$$

$$= \phi^3 \hat{X}_t(K-3)$$

$$\vdots$$

$$= \phi^K \hat{X}_t(K-K)$$

$$= \phi^K \hat{X}_t(0)$$

$$= \phi^K X_t$$

$$\boxed{\text{As } K \to \infty, \quad \phi^K X_t \to 0}$$

LET'S CONSIDER AR(1) AND CALCULATE THE EXPECTED VALUE

$$X_t = \phi X_{t-1} + \varepsilon_t$$

$$E(X_t) = E(\phi X_{t-1} + \varepsilon_t)$$

$$\underbrace{E(X_t)}_{\mu_x} = \phi \underbrace{E(X_{t-1})}_{\mu_x} + \underbrace{E(\varepsilon_t)}_{=0}$$

MEAN IS CONSTANT BECAUSE OF STATIONARITY

$$\mu_x = \phi \mu_x + 0$$

$$\mu_x - \phi \mu_x = 0$$

$$(1-\phi)\mu_x = 0$$

$$\boxed{\mu_x = \frac{0}{1-\phi} = 0}$$

PROPERTY: X_t IS MEAN-REVERTING

→ FORECAST ERROR

$$e_t(K) = X_{t+K} - \hat{X}_t(K)$$

$$= \phi\left(X_{t+K-1} + \varepsilon_{t+K}\right) - \phi\,\hat{X}_t(K-1)$$

$$= \underbrace{\phi\left(X_{t+K-1} - \hat{X}_t(K-1)\right)}_{e_t(K-1)} + \varepsilon_{t+K}$$

$$= \phi^K e_t(0) + \varepsilon_{t+K} + \varepsilon_{t+K-1} + \varepsilon_{t+K-2} + \dots$$

$$\mathrm{Var}(e_t(K)) =$$

$$\underbrace{\phi^K \varepsilon_t}_{=0} + \varepsilon_{t+K} + \varepsilon_{t+K-1} + \dots$$

$$\underbrace{\mathrm{Var}(\varepsilon_{t+K})}_{\sigma_\varepsilon^2} + \underbrace{\mathrm{Var}(\varepsilon_{t+K-1})}_{\sigma_\varepsilon^2} + \underbrace{\dots}_{\sigma_\varepsilon^2}$$