# Exercise 6.1

CONSIDER AR(1) MODEL

$$X_t = \phi X_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$

1) FIND THE 1-STEP AHEAD FORECAST OF $X_{t+1}$.
FIND THE VARIANCE OF THE 1-STEP AHEAD FORECAST.

WE KNOW $X_t = \phi X_{t-1} + \varepsilon_t$

SO, BY APPEALING TO THE ASSUMPTION OF PARAMETER STABILITY, THIS EQUATION WILL HOLD FOR TIMES $t+1$, $t+2$, AND SO ON.

$$X_{t+1} = \phi X_t + \varepsilon_{t+1}$$

$$X_{t+2} = \phi X_{t+1} + \varepsilon_{t+2}$$

$$X_{t+3} = \phi X_{t+2} + \varepsilon_{t+3}$$

$$\vdots$$

$$X_{t+K} = \phi X_{t+K-1} + \varepsilon_{t+K}$$

THE 1-STEP AHEAD FORECAST IS DEFINED BY THE FORMULA

$$\hat{X}_t(1) = E[X_{t+1} | F_t] = [X_{t+1}]$$

WE COMPUTE $\hat{X}_t(1)$

$$\hat{X}_t(1) = E[X_{t+1} | F_t] = [X_{t+1}]$$

$$= [\phi X_t + \varepsilon_{t+1}]$$

$$= [\phi X_t] + [\varepsilon_{t+1}]$$

$$= \phi \underbrace{[X_t]} + \underbrace{[\varepsilon_{t+1}]}$$

$$[\varepsilon_{t+1}] = E[\varepsilon_{t+1} | F_t] = 0$$

$\varepsilon_{t+1}$ IS A WHITE NOISE PROCESS AND IT IS INDEPENDENT OF THE PAST (HISTORY) $F_t$

$$[X_t] = E[X_t | F_t] = X_t$$

GIVEN THAT ALL INFORMATION IS KNOWN UP TO AND INCLUDING THAT AT TIME $t$ IS AVAILABLE, THE VALUE OF $X$ AT TIME $t$ IS KNOWN.

THEREFORE:

$$\hat{X}_t(1) = \phi X_t + 0$$

$$\hat{X}_t(1) = \phi X_t$$

THE ERROR OF 1-STEP AHEAD FORECAST IS:

$$e_t(1) = \underbrace{X_{t+1}} - \underbrace{\hat{X}_t(1)}$$

$$= \phi X_t + \varepsilon_{t+1} - \phi X_t$$

$$= \varepsilon_{t+1}$$

THE VARIANCE OF THE ERROR IS:

$$Var(e_t(1)) = Var(\varepsilon_{t+1}) = \sigma^2_\varepsilon$$

2) FIND THE 2-STEP AHEAD FORECAST OF $X_{t+2}$.
FIND THE VARIANCE OF THE 2-STEP AHEAD FORECAST.

WE CAN WRITE $\quad X_{t+2} = \phi X_{t+1} + \varepsilon_{t+2}$

THEN:

$$\hat{X}_t(2) = E[X_{t+2} | F_t] = [X_{t+2}]$$

$$= [\phi X_{t+1} + \varepsilon_{t+2}]$$

$$= \phi \underbrace{[X_{t+1}]} + \underbrace{[\varepsilon_{t+2}]}_{=0} =$$

THIS IS 1-STEP AHEAD FORECAST
$= \hat{X}_t(1) = \phi X_t$

$= 0$ BECAUSE $\varepsilon_{t+2}$ IS INDEPENDENT OF HISTORY $F_t$

$$= \phi \hat{X}_t(1) =$$

$$= \phi(\phi X_t) = \phi^2 X_t$$

SO $\quad \hat{X}_t(2) = \phi \hat{X}_t(1) = \phi^2 X_t$

THE ERROR OF 2-STEP AHEAD FORECAST IS

$$
\begin{aligned}
e_t(2) &= X_{t+2} - \hat{X}_t(2) \\
&= \phi X_{t+1} + \varepsilon_{t+2} - \phi \hat{X}_t(1) \\
&= \phi \underbrace{\left(X_{t+1} - \hat{X}_t(1)\right)}_{= e_t(1) = \varepsilon_{t+1}} + \varepsilon_{t+2} \\
&= \phi \varepsilon_{t+1} + \varepsilon_{t+2}
\end{aligned}
$$

THE VARIANCE OF THE ERROR IS:

$$
\begin{aligned}
Var(e_t(2)) &= Var(\phi \varepsilon_{t+1} + \varepsilon_{t+2}) \\
&= Var(\phi \varepsilon_{t+1}) + Var(\varepsilon_{t+2}) \\
&= \phi^2 \underbrace{Var(\varepsilon_{t+1})}_{= \sigma_\varepsilon^2} + \underbrace{Var(\varepsilon_{t+2})}_{= \sigma_\varepsilon^2} \\
&= \phi^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^2 \\
&= (\phi^2 + 1) \sigma_\varepsilon^2
\end{aligned}
$$

3) FIND THE K-STEP AHEAD FORECAST OF $X_{t+K}$.
FIND THE VARIANCE OF THE K-STEP AHEAD FORECAST.
COMMENT ON PROPERTIES OF THIS FORECAST
WHEN K INCREASES.

$$\hat{X}_t(K) = E[X_{t+K} | F_t] = [X_{t+K}]$$

$$= [\phi X_{t+K-1} + \varepsilon_{t+K}]$$

$$= \phi \underbrace{[X_{t+K-1}]}_{\text{THIS IS THE K-1-STEP AHEAD FORECAST}} + \underbrace{[\varepsilon_{t+K}]}_{=0}$$

$$= \phi \hat{X}_t(K-1)$$

$$= \phi (\phi \hat{X}_t(K-2))$$

$$= \phi^2 \hat{X}_t(K-2)$$

$$= \phi^2 (\phi \hat{X}_t(K-3))$$

$$= \phi^3 \hat{X}_t(K-3)$$

$$\vdots$$

$$= \phi^K \hat{X}_t(0)$$

$$= \phi^K \hat{X}_t$$

WE SEE THAT

$$\hat{X}_t(K) = \phi^K X_t \to 0$$

AS K INCREASES.

THIS IS THE MEAN - REVERSION PROPERTY SINCE $E(X_t) = 0$

$$\to E(X_t) = E(\phi X_{t-1}) + E(\varepsilon_t)$$

$$\underbrace{E(X_t)}_{\mu} = \phi \underbrace{E(X_{t-1})}_{\mu} + \underbrace{E(\varepsilon_t)}_{=0}$$

$$\mu - \phi\mu = 0$$

$$\mu = \frac{0}{1-\phi} = 0$$

# Problem 6.2

$X_t$ IS A NON-STATIONARY SERIES

$Z_t = X_t - X_{t-1}$ IS THE STATIONARY FIRST DIFFERENCE WHICH FITS AN AR(1) MODEL

$$Z_t = 2 + 0.4\, Z_{t-1} + \varepsilon_t$$

- FORECAST $X_{t+1}$ AND $X_{t+2}$

FIRSTLY WE COMPUTE THE 1-STEP AHEAD FORECAST OF $Z_t$

$$\hat{Z}_t(1) = E[Z_{t+1} \mid F_t] = [Z_{t+1}]$$

SINCE PARAMETER CONSTANCY OVER TIME IS ASSUMED, THEN $Z_{t+1} = 2 + 0.4\, Z_t + \varepsilon_{t+1}$

So
$$\hat{Z}_t(1) = [Z_{t+1}]$$
$$= [2 + 0.4\, Z_t + \varepsilon_{t+1}]$$
$$= 2 + 0.4 \underbrace{[Z_t]}_{Z_t} + \underbrace{[\varepsilon_{t+1}]}_{= 0 = E[\varepsilon_{t+1} \mid F_t]}$$

we know Z at time t.

$$\hat{Z}_t(1) = 2 + 0.4\,\underbrace{(Z_t)}$$

$$= 2 + 0.4\,\overbrace{(X_t - X_{t-1})}$$

THE 2- STEP AHEAD FORECAST FOR $Z_t$ IS:

$$\hat{Z}_t(2) = [Z_{t+2}]$$

$$= [2 + 0.4\,Z_{t+1} + \varepsilon_{t+2}]$$

$$= 2 + 0.4\,[Z_{t+1}] + \underbrace{[\varepsilon_{t+2}]}_{=0}$$

THIS IS THE
1- STEP AHEAD FORECAST
$[Z_{t+1}] = \hat{Z}_t(1) = 2 + 0.4\,Z_t$

$$= 2 + 0.4\,\hat{Z}_t(1)$$

$$= 2 + 0.4\,(2 + 0.4\,Z_t)$$

$$= 2 + 0.8 + 0.16\,Z_t$$

$$= 2.8 + 0.16\,\underbrace{Z_t}$$

$$= 2.8 + 0.16\,\overbrace{(X_t - X_{t-1})}$$

THEN WE COMPUTE $\hat{X}_t(1)$ USING THE EQUALITY

$$Z_{t+1} = X_{t+1} - X_t \quad \Leftrightarrow$$

$$X_{t+1} = X_t + Z_{t+1}$$

So

$$\hat{X}_t(1) = [X_{t+1}]$$

$$= [X_t + Z_{t+1}]$$

$$= \underbrace{[X_t]}_{= X_t} + \underbrace{[Z_{t+1}]}_{\hat{Z}_t(1)}$$

$$= X_t + \hat{Z}_t(1)$$

$$= X_t + 2 + 0.4(X_t - X_{t-1})$$

$$= X_t + 2 + 0.4 X_t - 0.4 X_{t-1}$$

$$= 2 + 1.4 X_t - 0.4 X_{t-2}$$

WE NOW COMPUTE $\hat{X}_t(2)$

NOTE THAT
$X_{t+2} = X_{t+1} + Z_{t+2}$

$$\hat{X}_t(2) = [X_{t+2}]$$

$$= [X_{t+1} + Z_{t+2}]$$

$$= [X_{t+1}] + [Z_{t+2}]$$

$$= \hat{X}_t(1) + \hat{Z}_t(2)$$

$$= \{2 + 1.4X_t - 0.4X_{t-1}\} + \{2.8 + 0.16(X_t - X_{t-1})\}$$

$$= 2 + 1.4X_t - 0.4X_{t-1} + 2.8 + 0.16X_t - 0.16X_{t-1}$$

$$= 4.8 + 1.56X_t - 0.56X_{t-1}$$

# Exercise 6.3

CONSIDER MA(2) MODEL

$$X_t = c_0 + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

where $\varepsilon_t$ is $WN(0, \sigma_\varepsilon^2)$

1) FIND 1-STEP, 2-STEP, 3-STEP AHEAD FORECAST.

BECAUSE OF THE ASSUMPTION OF PARAMETER STABILITY, IF THIS RELATIONSHIP HOLDS FOR THE SERIES X AT TIME t, IT IS ALSO ASSUMED TO HOLD FOR X AT TIME t+1, t+2, ...

$$X_{t+1} = c_0 + \varepsilon_{t+1} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1}$$

$$X_{t+2} = c_0 + \varepsilon_{t+2} + \theta_1 \varepsilon_{t+1} + \theta_2 \varepsilon_t$$

$$X_{t+3} = c_0 + \varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1}$$

$$\hat{X}_t(1) = E[X_{t+1} | F_t] = [X_{t+1}]$$

$$= [c_0 + \varepsilon_{t+1} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1}]$$

THE 1- STEP AHEAD FORECAST FOR $X$ IS GIVEN BY THE LINEAR COMBINATIONS OF THE DISTURBANCE TERMS. NOTE THAT IT WOULD NOT BE APPROPRIATE TO SET THE VALUES OF THESE DISTURBANCE TERMS TO THEIR UNCONDITIONAL MEAN OF ZERO. THIS ARISES BECAUSE IT IS THE CONDITIONAL EXPECTATION OF THEIR VALUES THAT IS OF INTEREST. GIVEN THAT ALL INFORMATION IS KNOWN UP TO INCLUDING AT THE TIME $t$ IS AVAILABLE, THE VALUES OF THE ERROR TERMS UP TO TIME $t$ ARE KNOWN. BUT $U_{t+1}$ IS NOT KNOWN AT TIME $t$ AND THEREFORE $E(U_{t+1} | F_t) = 0$.

WE CAN WRITE:

$$\hat{X}_t(1) = [c_0 + \varepsilon_{t+1} + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1}]$$

$$= \underbrace{[c_0]}_{= c_0} + \underbrace{[\varepsilon_{t+1}]}_{= E[\varepsilon_{t+1} | F_t] = 0} + \theta_1 \underbrace{[\varepsilon_t]}_{\varepsilon_t} + \theta_2 \underbrace{[\varepsilon_{t-1}]}_{= \varepsilon_{t-1}}$$

SO $\hat{X}_t(1) = c_0 + \theta_1 \varepsilon_t + \theta_2 \varepsilon_{t-1}$

$$\hat{X}_t(2) = [c_0 + \varepsilon_{t+2} + \theta_1 \varepsilon_{t+1} + \theta_2 \varepsilon_t]$$

$$= c_0 + \underbrace{[\varepsilon_{t+2}]}_{=0} + \theta_1 \underbrace{[\varepsilon_{t+1}]}_{=0} + \theta_2 \underbrace{[\varepsilon_t]}_{\varepsilon_t}$$

$$= c_0 + \theta_2 \varepsilon_t$$

$$\hat{X}_t(3) = [c_0 + \varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1}]$$

$$= c_0 + \underbrace{[\varepsilon_{t+3}]}_{=0} + \theta_1 \underbrace{[\varepsilon_{t+2}]}_{=0} + \theta_2 \underbrace{[\varepsilon_{t+1}]}_{=0}$$

$$= c_0$$

2) SHOW THAT THE FORECAST REVERTS TO THE MEAN AT STEP K

AS THE MA(2) PROCESS HAS A MEMORY OF ONLY TWO PERIODS, ALL FORECASTS THREE OR MORE STEP AHEAD COLLAPSE TO THE MEAN.

## 3) FIND THE FORECAST ERRORS.

- $e_t(1) = X_{t+1} - \hat{X}_t(1)$

$$= \{\cancel{c_0} + \mathcal{E}_{t+1} + \cancel{\theta_1 \mathcal{E}_t} + \cancel{\theta_2 \mathcal{E}_{t-1}}\} - \{\cancel{c_0} + \cancel{\theta_1 \mathcal{E}_t} + \cancel{\theta_2 \mathcal{E}_{t-1}}\} =$$

$$= \mathcal{E}_{t+1}$$

$$Var(e_t(1)) = Var(\mathcal{E}_{t+1}) = \sigma^2_{\mathcal{E}}$$

- $e_t(2) = X_{t+2} - \hat{X}_t(2)$

$$= \{\cancel{c_0} + \mathcal{E}_{t+2} + \theta_1 \mathcal{E}_{t+1} + \cancel{\theta_2 \mathcal{E}_t}\} - \{\cancel{c_0} - \cancel{\theta_2 \mathcal{E}_t}\} =$$

$$= \mathcal{E}_{t+2} + \theta_1 \mathcal{E}_{t+1}$$

$$Var(\mathcal{E}_t(2)) = Var(\mathcal{E}_{t+2} + \theta_1 \mathcal{E}_{t+1})$$

$$= \underbrace{Var(\mathcal{E}_{t+2})}_{\sigma^2_{\mathcal{E}}} + \theta_1^2 \underbrace{Var(\mathcal{E}_{t+1})}_{\sigma^2_{\mathcal{E}}}$$

$$= \sigma^2_{\mathcal{E}} + \theta_1^2 \sigma^2_{\mathcal{E}} = (1 + \theta_1^2)\sigma^2_{\mathcal{E}}$$

- $e_t(3) = X_{t+3} - \hat{X}_t(3) =$

$$= \{c_0 + \varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1}\} - \{c_0\}$$

$$= \varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1}$$

$$Var(e_t(3)) = Var(\varepsilon_{t+3} + \theta_1 \varepsilon_{t+2} + \theta_2 \varepsilon_{t+1})$$

$$= \underbrace{Var(\varepsilon_{t+3})}_{=\sigma_\varepsilon^2} + \theta_1^2 \underbrace{Var(\varepsilon_{t+2})}_{=\sigma_\varepsilon^2} + \theta_2^2 \underbrace{Var(\varepsilon_{t+1})}_{=\sigma_\varepsilon^2}$$

$$= \sigma_\varepsilon^2 + \theta_1^2 \sigma_\varepsilon^2 + \theta_2^2 \sigma_\varepsilon^2$$

$$= (1 + \theta_1^2 + \theta_2^2)\sigma_\varepsilon^2$$