Vectors & Matrices

Problem Sheet 9

1. We represent any value of the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ (where $a, b, c \in \mathbb{Q}$) as a column vector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{Q}^3 .$$

   (i) Find a matrix A that takes the vector representing $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ and maps it to the vector representing this value after multiplication by a factor of $1 - \sqrt[3]{2} + 2\sqrt[3]{4}$.

   (ii) Use Gauss-Jordan Inversion to find the inverse of A.

   (iii) Express the value

$$\frac{3 + 6\sqrt[3]{2} + 7\sqrt[3]{4}}{1 - \sqrt[3]{2} + 2\sqrt[3]{4}}$$

   in the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$, for some $a, b, c \in \mathbb{Q}$.

2. Consider the linear system $A\mathbf{x} = \mathbf{0}$, where A is a 3×3 matrix with at least one non-zero entry in its first column.

   (i) Show that if the solution set of this system can be expressed as a line $\mathbf{x} = \mathbf{p} + \lambda\mathbf{u}$ (for some fixed $\mathbf{p}$ and non-zero $\mathbf{u} \in \mathbb{R}^3$), then A is row equivalent to a matrix of the form

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

   for some $\alpha, \beta \in \mathbb{R}$.

   (ii) Show that if the solution set of this system can be expressed as a plane $\mathbf{x} \cdot \mathbf{n} = d$ (for some fixed non-zero $\mathbf{n} \in \mathbb{R}^3$ and scalar $d \in \mathbb{R}$), then A is row equivalent to a matrix of the form

$$\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

   for some $\alpha, \beta \in \mathbb{R}$.

3. Prove that if a matrix A is equal to its own inverse, then there exists a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{v}$ or $A\mathbf{v} = -\mathbf{v}$.