Topic: AUTOREGRESSIVE MODELS

Exercise 5.1

CONSIDER A STATIONARY AR(1) PROCESS:

$$X_t = \phi X_{t-1} + \varepsilon_t$$

where ε_t IS A $WN(0, \sigma^2_\varepsilon)$ AND $|\phi| < 1$

(i) PROVE THAT $E(X_t) = 0$

$$E(X_t) = E(\phi X_{t-1} + \varepsilon_t)$$

$$= E(\phi X_{t-1}) + \underbrace{E(\varepsilon_t)}_{=0}$$

$$= \phi E(X_{t-1})$$

ε_t IS A WHITE NOISE PROCESS WITH ZERO MEAN

SINCE $|\phi| < 1$, X_t IS A STATIONARY PROCESS. THEREFORE $E(X_t) = E(X_{t-1}) = \mu$ $\rightarrow$ THE MEAN DOES NOT DEPEND ON TIME t.

WE HAVE $\underbrace{E(X_t)}_{=\mu} = \phi \underbrace{E(X_{t-1})}_{=\mu}$

$$\mu = \phi \mu$$

$$\mu - \phi \mu = 0$$

$$(1-\phi)\mu = 0 \iff \mu = \frac{0}{1-\phi} = 0$$

(ii) PROVE THAT $Var(X_t) = \frac{\sigma_\varepsilon^2}{1-\phi^2}$

$$Var(X_t) = E\left[(X_t - E(X_t))^2\right]$$

WE KNOW THAT $E(X_t) = 0$; THEREFORE

$$Var(X_t) = E\left[(X_t - E(X_t))^2\right] =$$

$$= E\left[(X_t)^2\right] =$$

$$= E\left[(\phi X_{t-1} + \varepsilon_t)^2\right] =$$

$$= E\left[\phi^2 X_{t-1}^2 + \varepsilon_t^2 + 2\phi X_{t-1}\varepsilon_t\right] =$$

$$= \phi^2 \underbrace{E\left[(X_{t-1})^2\right]}_{} + \underbrace{E\left[\varepsilon_t^2\right]}_{=\sigma_\varepsilon^2} + 2\phi \underbrace{E\left[X_{t-1}\varepsilon_t\right]}_{=0}$$

THIS IS THE VARIANCE OF X_{t-1}

ε_t IS A WHITE NOISE PROCESS WITH ZERO MEAN AND VARIANCE σ_ε^2

BECAUSE FUTURE IS NOT CORRELATED WITH THE PAST

SINCE X_t IS A STATIONARY PROCESS, THE VARIANCE REMAINS CONSTANT AND INDEPENDENT OF TIME t ; SO $Var(X_t) = Var(X_{t-1}) = \sigma_y^2$

THEREFORE:

$$\underbrace{\mathrm{Var}(X_t)}_{=\sigma_X^2} = \phi^2 \underbrace{E\left[(X_{t-1})^2\right]}_{=\sigma_X^2} + \underbrace{E\left[\varepsilon_t^2\right]}_{=\sigma_\varepsilon^2} + 2\phi \underbrace{E\left[X_{t-1}\,\varepsilon_t\right]}_{=0}$$

$$\sigma_X^2 = \phi^2 \sigma_X^2 + \sigma_\varepsilon^2$$

$$\sigma_X^2 - \phi^2 \sigma_X^2 = \sigma_\varepsilon^2$$

$$(1-\phi^2)\,\sigma_X^2 = \sigma_\varepsilon^2$$

$$\sigma_X^2 = \frac{\sigma_\varepsilon^2}{1-\phi^2}$$

(iii) SHOW THAT AUTOCOVARIANCE FUNCTION IS

$$\gamma_K = \frac{\sigma^2_\varepsilon}{1-\phi^2}\,\phi^K \qquad K = 0, 1, 2, \dots$$

$$Cov(X_t, X_{t-K}) = E\left[\left(X_t - \underbrace{E(X_t)}_{=0}\right)\left(X_{t-K} - \underbrace{E(X_{t-K})}_{=0}\right)\right]$$

AS DERIVED IN POINT (i)

$$= E\left[X_t\, X_{t-K}\right]$$

$$= E\left[\left(\phi X_{t-1} + \varepsilon_t\right) X_{t-K}\right] =$$

$$= E\left[\phi X_{t-1} X_{t-K} + \varepsilon_t X_{t-K}\right]$$

$$= \phi\, \underbrace{E\left[X_{t-1} X_{t-K}\right]} + \underbrace{E\left[\varepsilon_t X_{t-K}\right]}_{=0}$$

THIS IS $Cov(X_{t-1}, X_{t-K})$.

$= \gamma_{K-1}$

BECAUSE FUTURE IS NOT CORRELATED WITH THE PAST

BECAUSE OF STATIONARITY:

$$\gamma_K = Cov(X_t, X_{t-1}) = E(X_t \cdot X_{t-K}) \;..\; \text{AND}$$

$$\gamma_{K-1} = E(X_{t-1} X_{t-K})$$

THEREFORE WE OBTAIN

$$\gamma_K = \phi\, \gamma_{K-1}$$

IF $\gamma_K = \phi \gamma_{K-1}$, THE WE CAN DEDUCE THAT

$\gamma_{K-1} = \phi \gamma_{K-2}$ AND $\gamma_{K-2} = \phi \gamma_{K-3}$ AND SO ON..

THEREFORE:

$$\gamma_K = \phi \gamma_{K-1}$$

$$\gamma_K = \phi (\phi \gamma_{K-2})$$

$$\gamma_K = \phi^2 \gamma_{K-2}$$

$$\gamma_K = \phi^2 (\phi \gamma_{K-3})$$

$$\gamma_K = \phi^3 \gamma_{K-3}$$

$$\vdots$$

$$\gamma_K = \phi^K \gamma_0$$

WHERE γ_0 IS THE VARIANCE OF X_t

$\gamma_0 = \dfrac{\sigma^2}{1-\phi^2}$ (AS PROVED IN ii)

WE CAN CONCLUDE THAT THE AUTOCOVARIANCE FUNCTION IS

$$\gamma_K = \phi^K \gamma_0$$

$$\gamma_K = \phi^K \frac{\sigma^2}{1-\phi^2} \qquad \text{FOR } K = 0, 1, 2, \ldots$$

SHOW THAT AUTOCORRELATION FUNCTION IS

$$\rho_K = \phi^K \qquad K = 0, 1, 2, \dots$$

$$\rho_K = \mathrm{Corr}(X_t, X_{t-K}) =$$

$$= \frac{\mathrm{Cov}(X_t, X_{t-K})}{\sqrt{\mathrm{Var}(X_t)}\sqrt{\mathrm{Var}(X_{t-K})}} =$$

BECAUSE X_t IS A STATIONARY PROCESS

$\mathrm{Var}(X_t) = \mathrm{Var}(X_{t-K})$

THEREFORE $\sqrt{\mathrm{Var}(X_t)} \cdot \sqrt{\mathrm{Var}(X_{t-K})} = \mathrm{Var}(X_t)$

$$= \frac{\mathrm{Cov}(X_t, X_{t-K})}{\mathrm{Var}(X_t)} =$$

$$= \frac{\gamma_K}{\gamma_0}$$

WE KNOW THAT $\gamma_K = \phi^K \gamma_0$

$$\rho_K = \frac{\phi^K \gamma_0}{\gamma_0} = \phi^K \qquad \text{for } K = 0, 1, 2, \dots$$

$\longrightarrow$ if $K = 0$ THEN $\rho_0 = \phi^0 = 1$

if $K = 1$ THEN $\rho_1 = \phi^1 = \phi$

if $K = 2$ THEN $\rho_2 = \phi^2$

$\vdots$

TOPIC : INFORMATION CRITERIA

INFORMATION CRITERIA IS GENERALLY USED FOR ARMA MODEL SELECTION.

INFORMATION CRITERIA EMBODY TWO FACTORS:

- A TERM WHICH IS A FUNCTION OF THE RESIDUAL SUM OF SQUARES (RSS)
- SOME PENALTY FOR THE LOSS OF DEGREES OF FREEDOM FROM ADDING EXTRA PARAMETERS.

SO ADDING A NEW ADDITIONAL LAG TO THE MODEL WILL THAT TWO COMPETING EFFECTS: RSS WILL FALL BUT THE VALUE OF PENALTY TERM WILL INCREASE.

THE OBJECT IS TO CHOOSE THE NUMBER OF PARAMETERS WHICH MINIMISES THE VALUE OF THE INFORMATION CRITERIA

THE TWO MOST POPULAR INFORMATION CRITERIA ARE:

$$AIC = \ln(\hat{\sigma}^2) + 2\frac{K}{T}$$

$$SBIC = \ln(\hat{\sigma}^2) + \frac{K}{T}\ln(T)$$

where $\hat{\sigma}^2$ IS THE RESIDUAL VARIANCE
K IS THE NUMBER OF PARAMETERS
T IS THE SAMPLE SIZE

→ Which criterion should be preferred if they suggest different model orders?

- SBIC IS STRONGLY CONSISTENT (BUT INEFFICIENT)
- AIC IS NOT CONSISTENT, BUT IS GENERALLY MORE EFFICIENT

THEN, OVERALL NO CRITERION IS DEFINITELY SUPERIOR TO OTHERS.

Exercise 5.2

CONSIDER THE DATA IN THE FILE "EXERCISE 5" AND FIT AN APPROPRIATE ARMA MODEL

ACF AND PACF ANALYSIS SHOW THAT WE CAN FIT EITHER AR(1) MODEL OR MA(3) MODEL.

ACF AND PACF DON'T PROVIDE A GOOD INDICATION FOR THE SELECTION OF ORDER p, q FOR FITTING ARMA(p, q) MODEL.

THEREFORE WE NEED TO USE INFORMATION CRITERIA. IN ORDER TO FIND THE BEST MODEL, WE NEED TO FIT DIFFERENT MODELS AND CHECK WHICH ONE MINIMISES AIC OR SBIC.

WE ESTIMATE DIFFERENT MODELS IN EVIEWS AND WE REPORT THE VALUES OF AIC AND SBIC CRITERIA IN THE FOLLOWING TABLE:

model	AIC	SBIC
AR (1)	8.738	8.789
MA (3)	8.772	8.873
ARMA (1, 1)	8.748	8.824

please note that:
WE DECIDED TO STOP TO ARMA (1, 1) BECAUSE THE MOVING AVERAGE COEFFICIENT IS NOT SIGNIFICANT. THIS INDICATES THE MODEL AR (1) IS PREFERABLE OVER AN ARMA (1, 1). FOR THIS REASON, IT DOES NOT MAKE SENSE TO CONTINUE WITH OTHER MODELS (e.g. ARMA (2, 1) or ARMA (1, 2))

Conclusion:

AIC CRITERION SUGGESTS AR (1) IS THE BEST FIT

SBIC CRITERION SUGGESTS AR (1) IS THE BEST FIT

$$AR(1): X_t = c + \phi X_{t-1} + \varepsilon_t$$

To estimate the model in Eviews

QUICK → ESTIMATE EQUATION

series 01 c AR(1)

options → ARMA METHOD: CLS → OK

THE ESTIMATION OUTPUT SHOWS:

$$X_t = 498.18 + 0.61\, X_{t-1} + \varepsilon_t$$

AR(1) COEFFICIENT $\phi = 0.61$ IS SIGNIFICANT. THE P-VALUE IS 0.0000 WHICH IS LOWER THAN 5% SIGNIFICANCE VALUE SO THE COEFFICIENT IS STATISTICALLY SIGNIFICANT.

→ LET'S CHECK IF RESIDUALS ARE CORRELATED

wiew → residual diagnostics
↓
correlogram Q-statistics

THE CORRELOGRAM SHOWS THAT RESIDUALS ARE NOT CORRELATED SO THE MODEL FITS WELL

→ forecasting:

THE FORECAST GRAPH SHOWS THAT WHEN THE STEP K INCREASES, THE FORECAST REVERTS TO THE MEAN WHICH IS ABOUT 500, AS IT SHOULD BE ACCORDING THE THEORY.

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recap

EViews steps FOR AIC / SBIC

- TO PLOT THE CORRELOGRAM

  double click on the series
  ↳ View → correlogram → lags to include 5/6

- AIC AND SBIC ARE OBTAINED IN THE ESTIMATION OUTPUT.
  ESTIMATE THE THREE FOLLOWING MODEL

  quick → estimate equation.

choose one of the three models

series01 c AR(1)
series01 c MA(1) MA(2) MA(3)
series01 c AR(1) MA(1)

options → ARMA METHOD : CLS