Exercise 3.3 (continued from last tutorial)

$$MA(1): \quad X_t = \varepsilon_t + \theta \varepsilon_{t-1}$$

$$E(X_t) = 0$$

$$Var(X_t) = \sigma_\varepsilon^2 (1 + \theta^2)$$

$$\Longrightarrow \quad Cov(X_t, X_{t-K}) = ?$$

$$\downarrow$$

$$E\Big[\big(X_t - \underbrace{E(X_t)}_{=0}\big)\big(X_{t-K} - \underbrace{E(X_{t-K})}_{=0}\big)\Big] =$$

because X_t is stationary

$$= E(X_t \, X_{t-K})$$

$$= E\Big[\underbrace{(\varepsilon_t + \theta \varepsilon_{t-1})}_{MA(1)} \underbrace{(\varepsilon_{t-K} + \theta \varepsilon_{t-K-1})}_{MA(1)}\Big]$$

$$= E\left(\varepsilon_t \varepsilon_{t-K} + \theta \varepsilon_t \varepsilon_{t-K-1} + \theta \varepsilon_{t-1} \varepsilon_{t-K} + \theta^2 \varepsilon_{t-1} \varepsilon_{t-K-1}\right)$$

$$\boxed{\text{IF } K = 1}$$

$\boxed{\text{IF } K=1}$

$$= \underbrace{E(\varepsilon_t \varepsilon_{t-1})}_{=0} + \theta \underbrace{E(\varepsilon_t \varepsilon_{t-2})}_{=0} + \theta \underbrace{E(\varepsilon_{t-1} \varepsilon_{t-1})} + \theta^2 \underbrace{E(\varepsilon_{t-1} \varepsilon_{t-2})}_{=0}$$

$Cov(\varepsilon_t, \varepsilon_{t-1})$

$\downarrow$

WE KNOW ε_t IS WHITE NOISE

$$\left.\begin{array}{l} E(\varepsilon_t) = 0 \\ Var(\varepsilon_t) = \sigma_\varepsilon^2 \\ Cor(\varepsilon_t, \varepsilon_{t-k}) = 0 \end{array}\right\} \text{independent of time}$$

$$E(\varepsilon_{t-1}, \varepsilon_{t-1}) =$$

$$= E(\varepsilon_{t-1}^2) = Var(\varepsilon_{t-1}) = \sigma_\varepsilon^2$$

$$= 0 + 0 + \theta\sigma_\varepsilon^2 + 0$$

$$\Rightarrow Cov(X_t, X_{t-1}) = \theta\sigma_\varepsilon^2$$

$\boxed{\text{IF } K = 2}$ $\boxed{\text{IF } K \geq 2}$

$$= \underbrace{E(\varepsilon_t \varepsilon_{t-2})}_{=0} + \theta \underbrace{E(\varepsilon_t \varepsilon_{t-3})}_{=0} + \theta \underbrace{E(\varepsilon_{t-1}, \varepsilon_{t-2})}_{=0} + \theta^2 \underbrace{E(\varepsilon_{t-1}, \varepsilon_{t-3})}_{=0}$$

$\Rightarrow \operatorname{Cov}(X_t, X_{t-2}) = 0$

FOR ANY $K \geq 2 \rightarrow \operatorname{Cov}(X_t, X_{t-K}) = 0$

IN GENERAL:

GIVEN MA(q), THE $\operatorname{Cov}(X_t, X_{t-K}) = 0$ WHEN $K > q$

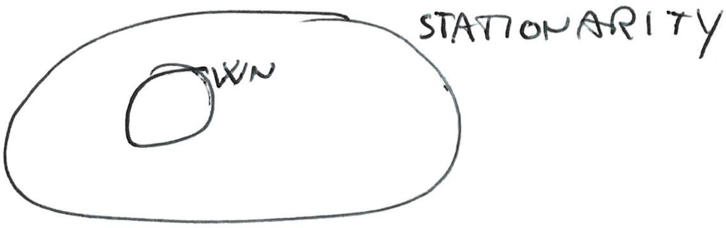
STATIONARITY (def):

X_t IS SAID TO BE STATIONARY IF:

$E(X_t) = \mu$ (independent of t)

$Var(X_t) = \sigma^2$ (")

$Cor(X_t, X_{t-k}) = \gamma_k$ (does not depend on t but on k)



MA(1) $X_t = \varepsilon_t + \theta \varepsilon_{t-1}$ where ε_t is WN

↳ MA IS STATIONARY

AR → AUTOREGRESSIVE MODEL

$$AR(1) = X_t = \underbrace{c_0}_{\text{constant}} + \phi_1 X_{t-1} + \varepsilon_t$$

$$AR(1) = X_t = \phi_1 X_{t-1} + \varepsilon_t$$

X_t is stationary if $|\phi| < 1$

AR (2): $X_t = c_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$

conditions for stationarity
$$\begin{cases} \phi_2 + \phi_1 < 1 \\ \phi_2 - \phi_1 < 1 \\ -1 < \phi_2 < +1 \end{cases}$$

Exercise 4.1

a) $X_t = -10 + 0.3 X_{t-1} + 0.5 X_{t-2} + \varepsilon_t$

$\Rightarrow$ $\phi_1 = 0.3$, $\phi_2 = 0.5$ $\longrightarrow$

$\phi_2 + \phi_1 = 0.8 < 1$ ✓

$\phi_2 - \phi_1 = 0.2 < 1$ ✓

$-1 < \phi_2 < 1$ ✓

WE HAVE STATIONARITY

b) $X_t = 35 + 0.3 X_{t-1} + 0.7 X_{t-2} + \varepsilon_t$

$\Rightarrow$ $\phi_1 = 0.3$, $\phi_2 = 0.7$ $\longrightarrow$

$\phi_1 + \phi_2 = 1$ ✗

$\phi_2 - \phi_1 = 0.4 < 1$ ✓

$-1 < \phi_2 < 1$ ✓

WE DO <u>NOT</u> HAVE STATIONARITY

Exercise 4.3

i) $H_0 : \rho_k = 0 \rightarrow$ series is a white noise

$H_1 : \rho_k \neq 0$

where $k = 1, \dots, 12$

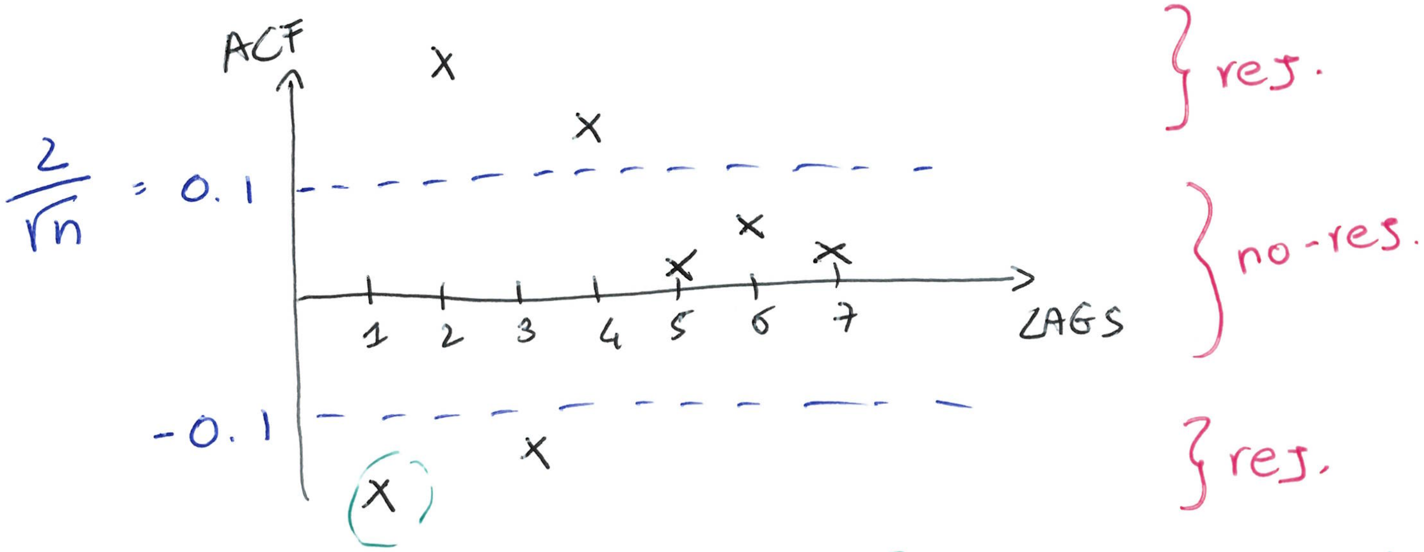
rej. rule: $\rightarrow$ reject H_0 if $|\hat{\rho}_k| > \frac{2}{\sqrt{n}}$

where $\frac{2}{\sqrt{n}} = \frac{2}{\sqrt{400}} = 0.1$

t-stat: $\frac{\hat{\rho} - \rho_{H0}}{SE(\hat{\rho})} = \frac{\hat{\rho} - \rho_{H0}}{\sigma / \sqrt{n}}$ $\rightarrow = 0$

UNDER H_0 $\rho \rightarrow N(0,1)$

$\sigma = 1$



$\hookrightarrow |\hat{\rho}_1| = 0.24 > \frac{2}{\sqrt{400}} = 0.1 \rightarrow$ rej H_0

ii) → AR (p) HOW TO CHOOSE p?

iii) → MA (q) HOW TO CHOOSE q?

ACF → best MA model → q

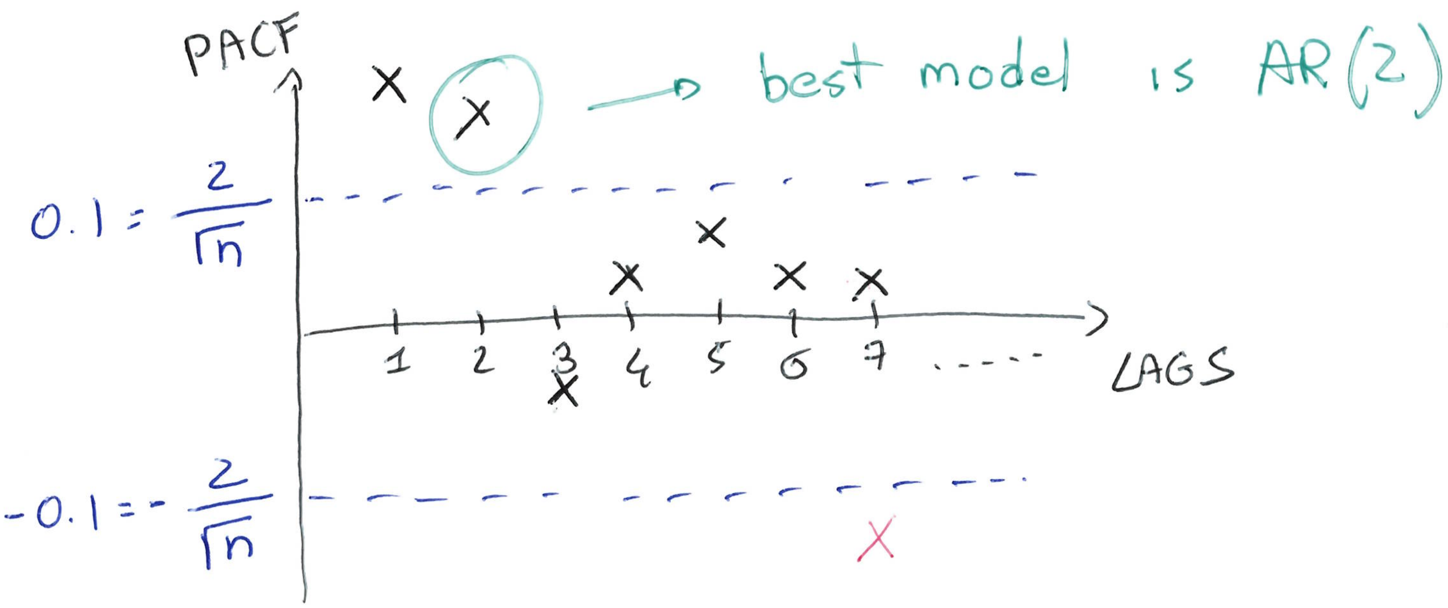
PACF → best AR Model → p

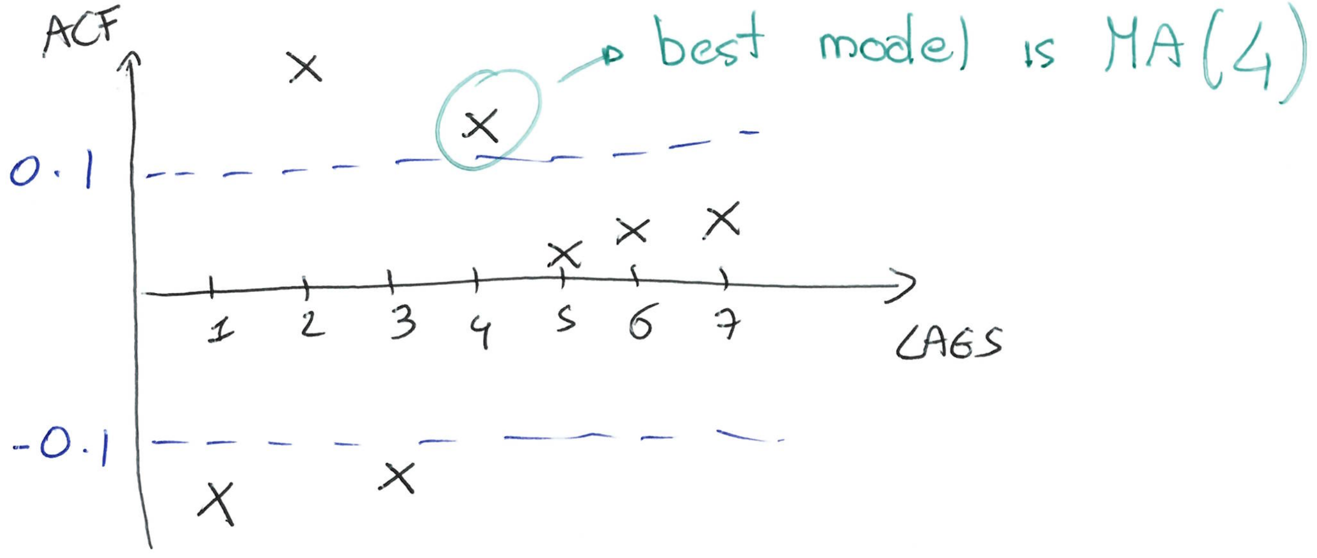
↓

TEST EACH PACF VALUE AND SEE IF THEY ARE SIGNIFICANT OR NOT

↓

p = LAG OF THE LAST SIGNIFICANT CORR. COEFFICIENT





(IV) ACF AND PACF SUGGEST THAT THE BEST MODELS ARE: AR(2) AND MA(4).

BETWEEN THE TWO MODELS, WE PREFER TO USE AR(2)

↳ LESS COEFFICIENTS TO ESTIMATE

AR(2): $X_t = C_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$

MA(4): $X_t = C_0 + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4} + \varepsilon_t$